THE MERCHANDISING MATHEMATICIAN MODEL. STOCHASTIC DEMAND AND SUPPLY

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Abstract

A simple model of a buying-selling cycle is proposed. The model comprises two moves: a rational buying and a random selling. The notion of a profit intensity is introduced. Supply and demand curves and geometrical interpretation are discussed in this context.

1 INTRODUCTION

The very aim of any conscious and rational economic activity is optimization of the profit in given economic conditions and, usually, during definite intervals. The interval is chosen so that it contains a certain characteristic economic cycle (e.g. one year, a season, an insurance period or a contract date). Of course, often it is possible and reasonable to make prognosis for a distant future of an undertaking by extrapolation from the already known facts. The quantitative description of an undertaking is extremely difficult when the time of duration of the intervals in question is itself a random variable (denoted by τ in the following). The profit gained during the specific period, described as a function of τ, becomes also a random variable and as that does not measure the quality of the undertaking. To investigate activities that have different periods of duration we define, following the queuing theory Billingsley (1979), the **profit intensity** as a measure of this economic category. An acceptable definition of the profit must provide us with an additive function. It seems that the proposed interval interest rate notion leads to consistent results.

2 THE PROFIT INTENSITY

Let t, υ_t and $\upsilon_{t+\tau}$ denote the beginning of an interval of the duration τ, the value of the undertaking (asset) at the beginning and at the end of the interval, respectively. We define the **logarithmic rate of return** $r_{t,t+\tau}$ as

$$r_{t,t+\tau} \equiv \ln\left(\frac{\upsilon_{t+\tau}}{\upsilon_t}\right). \tag{1}$$

Let the expectation value of the random variable ξ in one cycle (buying-selling or vice versa) be denoted by $E(\xi)$. If $E(r_{t,t+\tau})$ and $E(\tau)$ are finite then we define the **profit intensity** for one cycle ρ_t Piotrowski(1999) as

$$\rho_t \equiv \frac{E(r_{t,t+\tau})}{E(\tau)}. \tag{2}$$

This definition is an immediate consequence of the Wald identity Resnick(1998):

$$E(S_{\tau'}) = E(X_1)\,E(\tau')\ , \tag{3}$$

where $S_{\tau'} \equiv X_1 + \ldots + X_{\tau'}$ is the sum of τ' equally distributed random variables $X_k,\ k = 1, \ldots, \tau'$ and τ' is the stopping time, Billingsley (1979) and Resnick (1998). It is obvious that the profit intensity we have defined in the Equation (2) is just the $E(X_1)$ from the Wald identity, Equation (3). The expected profit is the left hand side of the Wald identity. If we are interested in the profit expected in a time unit we have, according to Wald, divide the expected profit by the expectation value of the stopping time, so we get the Equation (2). We can also calculate the variance of the profit intensity by using the proposition 10.14.4 from the reference Resnick (1998):

$$E\left(\left(S_{\tau'} - \tau' E\left(X_1\right)\right)^2\right) = E\left(\tau'\right) Var\left(X_1\right). \tag{4}$$

Of course, our definition of the profit intensity is applicable also in more general cases when the random variables X_i are correlated or have different distributions.

The profit expected after an arbitrary time interval, say $[0, T]$ is given by

$$\rho_{0,T} \equiv \int_0^T \rho_t dt\ . \tag{5}$$

The proposed definition of the profit intensity is a convenient starting point for the consideration of the proposed below model. Relations to the commonly used measures of profits (returns) can be easily obtained by simple algebraic manipulations.

3 THE MERCHANDISING MATHEMATICIAN MODEL

Let us consider the simplest possible market event of exchanging two goods which we would call the asset and the money and denote by Θ and \$, respectively. The proposed model comprises two moves. First move consist in a rational buying of the asset Θ (exchanging \$ for Θ). The meaning of the adjective rational will be explained below. The second move consist in a random (immediate) selling of the purchased amount of the asset Θ (exchanging Θ for \$). Note that the order of these transactions can be reversed and, in fact, is conventional. Let V_Θ and $V_\$$ denote some given amounts of the asset and the money, respectively. If at some time t the assets are exchanged in the proportion $V_\$: V_\Theta$ than we call the number

$$p_t \equiv \ln\left(V_\$\right) - \ln\left(V_\Theta\right) \tag{6}$$

the logarithmic quotation for the asset Θ. If the trader buys some amount of the asset Θ at the quotation p_{t_1} at the moment t_1 and sells it at the quotation p_{t_2} at the later moment t_2 then his profit (or more precise the logarithmic rate of return) will be equal to

$$r_{t_1,t_2} = p_{t_2} - p_{t_1}. \tag{7}$$

The logarithmic rate of return, contrary to p_t, does not depend on the choice of unit used to measure the assets in question. From the projective geometry point of view r_{t_1,t_2} is an invariant and p_t is not, cf the discussion of demand and supply curves in the Section 4.

The **merchandising mathematician model** (MM model) consists in what we call the rational purchase followed by a random selling of some asset Θ. The rational purchase is simply a purchase bound by a fixed **withdrawal price** $-a$ that is such a logarithmic quotation for the asset Θ, $-a$, above which the trader gives the buying up. The quotation method does not matter to the process of rational purchase. A random selling can be identified with the situation when the withdrawal price is set to $-\infty$ (the trader in question is always bidding against the rest of traders).

Let us suppose now that the model describes a stationary process, that is the probability density $\eta(p)$ of the random variable p (the logarithmic quotation) does not depend on time. Note that it is sufficient to know the logarithmic quotations up to arbitrary constant because what matters is the profit and profit is always a difference of quotations. This is analogous with the classical physics where only differences of the potential matter (cf Newton's gravity). Therefore we can suppose that expectation value of the random variable p is equal to zero, $E(p) = 0$. We shall also suppose that the market is large enough not to be influenced by our trader. Let the expression $[sentence]$ takes value 0 or 1 if the $sentence$ is false or true, respectively (Iverson convention), Graham, Knuth, and Patashnik (1994). The mean time of a random transaction (buying or selling, it is a matter of convention) will be denoted by θ. The value of θ is fixed in our model due to the stationarity assumption. Besides, to eliminate paradoxes (e.g. infinite profits during finite time spreads) θ should be greater than zero. Let x denote the probability that the rational purchase would not occur:

$$x \equiv E_\eta([p > -a]). \tag{8}$$

The expectation value of the rational purchase time of the asset Θ is equal to

$$\theta\left((1-x) + 2x(1-x) + 3x^2(1-x) + 4x^3(1-x) + \ldots\right). \tag{9}$$

The ratio of the expected duration of the whole buying-selling cycle and the expected time of a random reverse transaction is given by

$$\begin{array}{rcl} \frac{E_\eta(\tau)}{\theta} & = & 1+(1-x)\sum_{k=1}^{\infty} kx^{k-1} \\ & = & 1+(1-x)\frac{d}{dx}\sum_{k=0}^{\infty} x^k \\ & = & 1+(1-x)\frac{d}{dx}\frac{1}{1-x} = 1+\frac{1}{1-x} \end{array} . \tag{10}$$

Therefore the mean length of the whole cycle is given by

$$E_\eta(\tau) = \left(1+\left(E_\eta\left([p \le -a]\right)\right)^{-1}\right)\theta. \tag{11}$$

The logarithmic rate of return for the whole cycle is

$$r_{t,t+\tau} = -p_{\to\Theta} + p_{\Theta\to}, \tag{12}$$

where the random variable $p_{\to\Theta}$ (quotation at the moment of purchase) has the distribution restricted to the interval $(-\infty, -a]$:

$$\eta_{\to\Theta}(p) = \frac{[p \le -a]}{E_\eta\left([p \le -a]\right)}\,\eta(p)\,. \tag{13}$$

The random variable $p_{\Theta\to}$ (quotation at the moment of selling) has the probability density η, as the selling is at random. The expectation value of the of the profit after the whole cycle is

$$\rho_\eta(a) = \frac{-\int_{-\infty}^{-a} p\,\eta(p)\,dp}{1+\int_{-\infty}^{-a}\eta(p)\,dp}, \tag{14}$$

which follows from Equations (5) and (13). This function has very interesting properties (we will often drop the subscript η in the following text) stated as the Theorem 1.

THEOREM 1 *The maximal value of the function ρ , a_{max} , lies at a fixed point of ρ, that is fulfills the condition $\rho(a_{max}) = a_{max}$. Such a fixed point a_{max} exists and $a_{max} > 0$.*

Proof

The fixed point condition:

$$\frac{-\int_{-\infty}^{-a} p\,\eta(p)\,dp}{1+\int_{-\infty}^{-a}\eta(p)\,dp} = a \tag{15}$$

can be rewritten as :

$$a\left(1+\int_{-\infty}^{-a}\eta\left(p\right)dp\right)=-\int_{-\infty}^{-a}\left(p+a\right)\eta\left(p\right)dp+\int_{-\infty}^{-a}a\;\eta\left(p\right)dp. \tag{16}$$

This leads to

$$a=-\int_{-\infty}^{0}p\;\eta\left(p-a\right)dp. \tag{17}$$

The derivative of the righthand side of the Equation. (17) is equal to

$$-\int_{-\infty}^{-a}\eta\left(p\right)dp \tag{18}$$

and it is obvious that the righthand side of the Equation (17) is a non-increasing positive function of a that tends to 0 for $a\to\infty$. Remember that we have supposed that the expectation value of p is equal to 0 so that η cannot identically vanish for $p\leq 0$. To end the proof it sufficient to notice that the vanishing of the derivative of the function ρ :

$$a\;\eta\left(a\right)\left(1+\int_{-\infty}^{-a}\eta\left(p\right)dp\right)+\eta\left(a\right)\int_{-\infty}^{-a}p\;\eta\left(p\right)dp=0. \tag{19}$$

is exactly the fixed point condition (15) multiplied by $\eta\left(a\right)$. So for a non-vanishing $\eta\left(a\right)$ the proof ends here. If $\eta\left(a\right)$ vanishes then the derivative of ρ:

$$\frac{d\rho\left(a\right)}{da}=\frac{a\;\eta\left(a\right)}{1+\int_{-\infty}^{-a}\eta\left(p\right)dp}+\frac{\eta\left(a\right)\int_{-\infty}^{-a}p\;\eta\left(p\right)dp}{\left(1+\int_{-\infty}^{-a}\eta\left(p\right)dp\right)^{2}} \tag{20}$$

is non-negative in small vicinities of a and there is no extremum at a.

It might be useful to analyze an example here.

EXAMPLE 1 (normal distribution) *Let now $\eta\left(p,\sigma\right)$ be the standard normal probability density with the variance σ and expectation value $\hat{p}$ of a random variable p*

$$\eta\left(p,\sigma\right)\equiv\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(p-\hat{p}\right)^{2}}{2\sigma^{2}}\right) \tag{21}$$

In this case the expectation value of the profit during a whole cycle $\rho(a,\sigma)_{normal}$ *(we have explicitly shown the dependence on the variance* σ*) has a nice scaling property :*

$$\rho(a,\sigma)_{normal} = \sigma\rho(a,1)_{normal}, \tag{22}$$

and it is sufficient to work out the $\sigma = 1$ *case only. If this is the case we get the maximal expectation value of the profit for* $a = 0.27603$*. According to the Theorem 1. the maximal expected profit is also equal to* 0.27063*.*

It is worth to notice that the condition (17) clearly shows what the maximal possible profit is.

It tempting to claim that the function ρ is a contraction. In a general case this is not true. Simple inspection reveals that if the probability has a very narrow and high maximum then ρ is not a contraction in the vicinity of the maximum. But for any realistic probability density one can start at any value of a and by iteration wind up in the fixed point (Banach fixed point theorem). We skip the details because they are technical and unimportant for the conclusions of the paper.

4 DEMAND AND SUPPLY CURVES

The literature on economics including texts avoiding mathematical formalism abounds in graphs and diagrams presenting various demand and supply curves. For example, Blaug in Blaug (1985) quotes at least a hundred of such diagrams. This illustrates the importance the economists attach to them. Such approach is also possible in the MM model. Let us consider the functions

$$F_s \equiv E_{\eta_1}([\xi \leq x]) = \int_{-\infty}^{x} \eta_1(p)\,dp, \tag{23}$$

and

$$F_d \equiv E_{\eta_2}([\xi \geq x]) = \int_{x}^{\infty} \eta_2(p)\,dp, \tag{24}$$

where we have introduced two, in general case different, appropriate probability density η_1 and η_2. They may differ due to the existence of a monopoly, specific market regulations, taxes, cultural habits and so on. Let us recall that one can find two ways of presenting demand/supply curves in the literature. The first one (French) is based on the assumption that the demand is a function of prices and is usually referred to as the Cournot convention. The Anglo-Saxon literature prefers the Marshall convention with reversed roles of the coordinates. The demand (supply) is not always a monotonic function of prices (cf. the discussed below turning back of demand/supply curves) therefore the Marshall convention seems to be less convenient (one cannot use the notion of a function). The MM model with the price-like parameter x refers to the Cournot convention. So, for a fixed value x of the logarithm of the price of an asset Θ, the value of the supply function $F_s(x)$ is given by the probability of the purchase of a unit at the price e^x. The asset would be provided by everyone who is willing to sell it at the price e^x or lower than e^x. The function $F_d(x)$ can be interpreted in an analogous way. If we neglect the sources of possible differences between η_1 and η_2 and, in addition, suppose that at any fixed price there are no indifferent traders (that is everybody wants to sell or buy) then we can claim that

$$E_{\eta_1}\left([\zeta \leq x]\right) + E_{\eta_2}\left([\zeta > x]\right) = 1. \tag{25}$$

The differentiation of the Equation (25) leads to $\eta_1 = \eta_2$. Under these conditions the price e^y for which $E_{\eta_1}\left([\zeta \leq y]\right) = E_{\eta_2}\left([\zeta > y]\right) = \frac{1}{2}$ establishes the equilibrium price in the classical meaning. This simply means that this price the most frequent one. Recall that the MM model is scaled so that this price is $e^0 = 1$.

It would be instructive to analize the problem from the projective geometry point of view. In this approach the market is described in the $N-$dimensional real projective space, $\Re P^N$ that is $(N+1)-$dimensional vector space $\Re^{N+1}$ (one real coordinate for each asset) subjected to the equivalence relation $v \sim \lambda v$ for $v \in \Re^{N+1}$ and $\lambda \neq 0$. For example we identify all portfolios having assets in the same proportions. The actual values can be obtained by rescaling by λ. The details would be presented elsewhere. In this context separate profits gained by buying or selling are not invariant (coordinate free). The profit $r_{t,t+\tau}$ gained during the whole cycle is given by the logarithm of an appropriate anharmonic (cross) ratio, Courant and Robbins (1996), and is an invariant (e.g. its numerical value does not depend on units chosen to measure the assets). The anharmonic ratio for four points lying on a given line, $A,\ B,\ C,\ D$ is the double ratio of lengths of segments $\frac{AC}{AB} : \frac{DC}{DB}$ and is denoted by $[A, B, C, D]$. In our case the anharmonic ratio in question, $[\Theta, U_{\to\Theta}, U_{\Theta\to}, \$]$, concerns the pair of points:

$$U_{\to\Theta} \equiv (v, v \cdot e^{p_{\to\Theta}}, \ldots) \;\; and \; U_{\Theta\to} \equiv (w, w \cdot e^{p_{\Theta\to}}, \ldots) \tag{26}$$

and the pair $\Theta \; and \; \$$. The last pair results from the crossing of the hypersurfaces $\overline{\Theta}$ and $\overline{\$}$ corresponding to the portfolios consisting of only one asset Θ or $\$$, respectively and the line $U_{\to\Theta}U_{\Theta\to}$. The dots represent other coordinates (not necessary equal for both points). The line connecting $U_{\to\Theta}$ and $U_{\Theta\to}$ may be represented by the one-parameter family of vectors $u(\lambda)$ with $\mu-$coordinates given by

$$u_\mu(\lambda) \equiv \lambda \left(U_{\to\Theta}\right)_\mu + (1-\lambda) \cdot \left(U_{\Theta\to}\right)_\mu . \tag{27}$$

This implies that the values of λ corresponding to the points Θ and $\$$ are given by the conditions:

$$u_0(\lambda_\$) = \lambda_\$ \left(U_{\to\Theta}\right)_0 + (1-\lambda_\$) \cdot \left(U_{\Theta\to}\right)_0 = 0 \tag{28}$$

and

$$u_1(\lambda_\Theta) = \lambda_\Theta \left(U_{\to\Theta}\right)_1 + (1-\lambda_\Theta) \cdot \left(U_{\Theta\to}\right)_1 = 0. \tag{29}$$

Substitution of the Equation (26) leads to

$$\lambda_\$ = \frac{w}{w-v} \tag{30}$$

and

$$\lambda_\Theta = \frac{w e^{p_{\Theta\to}}}{w e^{p_{\Theta\to}} - v e^{p_{\to\Theta}}}. \tag{31}$$

The calculation of the logarithm of the cross ratio $[\Theta, U_{\to\Theta}, U_{\Theta\to}, \$]$ on the line $U_{\to\Theta}U_{\Theta\to}$ leads to

$$\begin{aligned} \ln[\Theta, U_{\to\Theta}, U_{\Theta\to}, \$] &= \ln\left[\tfrac{w e^{p_{\Theta\to}}}{w e^{p_{\Theta\to}} - v e^{p_{\to\Theta}}}, 1, 0, \tfrac{w}{w-v}\right] \\ &= \ln \tfrac{v \; w \; e^{p_{\Theta\to}}}{v \; e^{p_{\to\Theta}} \; w} = p_{\Theta\to} - p_{\to\Theta} \end{aligned} \tag{32}$$

which corresponds to the formula (7).

Contrary to the classical economics the balance in the MM model does not result in uniform quotations (prices) for the asset Θ but only in a stationarity of the supply and demand functions $E_{\eta_1}([\zeta \leq x])$ and $E_{\eta_2}([\zeta \geq x])$.

Therefore the MM model is not valid when the changes in the probabilities happens during periods shorter or of the order of the mean time transaction θ. Of course the presented above stochastic interpretation of the supply and demand remains valid in such situations. In addition we can consider piecewise decreasing functions F_s and F_d. Such generalization requires that these function cease to be probability distribution functions because their derivatives (probability densities) are not positive definite. This corresponds to the effect of *turning back* of the supply and demand curves what happens for work supplies and the Giffen goods, Stigler (1947). In the Marshall convention these curves loose the function property. In the Cournot convention these curves are diagrams of multivalued functions. In this way negative probability densities (Wigner function) gain interesting economics reason for the existence. Wigner functions emerged in the quantum theory, Feynman (1987). By a choice of stochastic process consistent with the MM model one can determine the dynamics of such a model, cf Blaquiere (1980). Therefore we suspect that the departure from the laws of supply and demand might be the first known example of a macroscopic reality governed by quantum-like rules. Such hypothetical quantum economics started with the evidence given by Robert Giffen in the British Parliament, Stigler (1947) would have earlier origin than the quantum physics. It should be noted here that from the quantum game theory point of view the Gauss distribution function is the only supply (demand) curve that fulfills the physical correspondence principle. The authors would devote a separate paper to this very interesting problem.

Let us note that the distribution functions allow for correct description of the famous Zeno paradoxes (when grains form a pile? when you start to be bald?) because the introduction of probabilities removes the original discontinuity. For example the problem of morally right prices: if the price is low (state 0) nobody wants to sell and if the price is high (state 1) everybody wants to sell. Without the probability theory we are not able to describe intermediate states which, in fact, are typical on the markets. Does it suggest that the MM model can also be applied to problems where there is a necessity of finding maximum (minimum) of a profit intensity like parameter?

5 CONCLUDING REMARKS

We have discussed the model where the trader fixes the maximal price he is willing to pay for the asset Θ and then after some time sells it at random. One can easily reverse the buying and selling strategies. If this is the case the formula (14) should be modified to:

$$\rho(b) = \frac{\int_b^\infty p\,\eta(p)\,dp}{1 + \int_b^\infty \eta(p)\,dp}, \tag{33}$$

where b denotes the minimal acceptable price of Θ (that is below which the trader gives up the selling).

It is interesting that the optimal behaviour of the trader consist in fixing the withdrawal price below the mean quotation so that the difference is exactly the profit expected during a mean buying-selling cycle. If he or she manages to do so then the optimal and stable position is reached and no further manoeuvring is necessary. So the best strategy is the self-consistent correction of the withdrawal price, cf Theorem 1. The existence of such a mechanism is highly required in dynamical markets where the distributions of quotations are continuously changing. Please note that if we set finite withdrawal prices for both type of transactions (buying and selling) the above simple recipe cease to work. One might ask if this suggests that the two-way transactions should be avoided? Or the only correct model is the one consisting in random buying followed by selling with fixed withdrawal price? One might suggest (suspect?) that the later case is the only one when it is possible to define the quotation distribution relatively to the subsequent selling. This might be compared with widely spread opinion among brokers that the moment of closing of a open position is much more important than the actual moment of opening this position (i.e. random buying). Of course the terms selling and buying are conventional: while selling the asset Θ one buys \$.

At the end we would like to note that the process of searching optimal solutions and fixed points are the key issues of contemporary mathematical economics, Debreu (1981). Such classical results as generalised Brouwer theorem, Kakutani (1941), and the Brown-Robinson iteration, Robinson (1951), are widely applied. The proposed MM model combines both ideas. The authors envisage the extension of the MM model to the randomized withdrawal price cases which might also generalise the results of Piotrowski and Sładkowski (2001), where thermodynamics of investors was considered and the temperature of portfolios was defined.

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