Arbitrage risk induced by transaction costs

(RePEc:sla:eakjkl:17 20-V-2003)

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Abstract

We discuss the time evolution of quotation of stocks and commodities and show that they form an Ising chain. We show that transaction costs induce arbitrage risk that usually is neglected. The full analysis of the portfolio theory is computationally complex but the latest development in quantum computation theory suggests that such a task can be performed on quantum computers.

PACS numbers: 02.50.Le, 03.67.Lx, 05.50.+q, 05.30.–d
Keywords: econophysics, financial markets, quantum computations, portfolio theory

1 Introduction

One can simply define arbitrage as an opportunity of making profit without any risk [1]. But this definition has one flaw: it neglects transaction costs. And any market activity involves costs (e.g. brokerage, taxes and others depending on the established rules). Therefore there is always some uncertainty and an arbitrageur cannot avoid risk. Below we will describe an extremely profitable manipulation of a one asset market that certainly fit this definition and show how brokerage can induce risk. The method allows to make maximal profits in a fixed interval $[0, k]$ (short-selling allows to make profits with arbitrary price changes). We will analyze the associated risk by introducing canonical arbitrage portfolios that admit Ising model like description. Investigation of such models is difficult from the computational point of view (the complication grows exponentially in k) but the latest development in quantum computation seems to pave the way for finding effective methods of solving the involved computational problems [2].

2 Profit from brokerage-free transactions

Standard descriptions of price movements, following Bachelier, use the formalism of diffusion theory and random variables. Such an approach to the problem involves the assumption of constancy of the parameters of model during the interval used for their estimation. Besides the use of the dispersion of the drifting logarithm of price as a measure of the risk might be questioned. A clairvoyant that knows the future evolution of prices would make profits from any price movement and it would be difficult to attribute any risk to her market activity. Rather, the level of erroneousness of our decisions concerning the portfolio structure should be used for that aim. Having this in mind we have proposed a dual formulation of the portfolio theory and market prices [3]-[6]. In this approach movements in prices are regarded as deterministic according to their historical record and the stochastic properties are attributed to portfolios. This enables us to use the formalism of information theory and thermodynamics. Due to the convenience of this approach we will adopt it in the current paper.

Consider a game against the Rest of the World (that is the whole market) that consist in alternate buying and selling of the same commodity. Let $h_m := \ln \frac{c_m}{c_{m-1}}$ denote logarithms of the prices dictated by the market of this commodity at successive moments $m = 1, 2, \ldots, k$. If the costs of transactions are zero (or negligible) then the player's profit (actually a loss because for future convenience we will fix the sign in (1) according to the standard physical convention) in the interval $[0,k]$ is given by

$$H(n_1,\ldots,n_k) := -\sum_{m=1}^{k} h_m n_m\,. \tag{1}$$

The elements of the sequence (n_m) take the value 0 or 1 if the player possesses money or the commodity at the moment m, respectively. The sequence (n_m) defines the player's strategy in a unique way and any $(n_1,\ldots,n_k)$ describes a pure strategy. Generalization to a more realistic situation where more commodities are available is trivial but besides complication of formulas is irrelevant to the conclusion and will not be considered here.

3 The thermodynamics of portfolios

Any mixed strategy can be parameterized in a unique way by 2^k weights $p_{n_1,\ldots,n_k}$ giving the contributions of pure strategies. Let us consider as equivalent all strategies that for a given price sequence $(h_1,\ldots,h_k)$ bring the same profit. We will call the equivalence classes of portfolios defined in this way the canonical portfolios. Any canonical portfolio has maximal information entropy

$$S_{(p_{n_1,\ldots,n_k})} := -E(\ln p_{n_1,\ldots,n_k}) \tag{2}$$

and can in a sense be regarded as an equilibrium state for portfolios in its class (the player rejects any superfluous from the market point of view) information. Claude Shannon's entropy $S_{(p_{n_1,\ldots,n_k})}$ rooted in cryptographic information theory is proportional to the minimal length of the compressed by the greedy Huffman algorithm computer code that contains information about the portfolio [7]. Therefore it seems to be reasonable to accept the risk incurred of investing in a given canonical portfolio as a measure of risk for the whole class it represents. The explicit form of a canonical portfolio can be found by the Lagrange multipliers method that leads to the requirement of vanishing of the following differential form:

$$dS_{(p_{n_1,\ldots,n_k})} - \beta\, dE(H(n_1,\ldots,n_k)) - \zeta\, d\sum_{n_1,\ldots,n_k=0}^{1} p_{n_1,\ldots,n_k} =$$

$$-\sum_{n_1,\ldots,n_k=0}^{1} \big(\ln p_{n_1,\ldots,n_k} + 1 + \beta\, H(n_1,\ldots,n_k) + \zeta\big)\, dp_{n_1,\ldots,n_k} = 0\,,$$

where β and ζ are Lagrange multipliers. It follows that the sum $\ln p_{n_1,\dots,n_k} + 1 + \beta\, H(n_1,\dots,n_k) + \zeta$ should vanish independently of the values of $dp_{n_1,\dots,n_k}$. Therefore the equation

$$\ln p_{n_1,\dots,n_k} + 1 + \beta\, H(n_1,\dots,n_k) + \zeta = 0$$

allows to find the dependence of the weights $p_{n_1,\dots,n_k}$ on the profits $-H(n_1,\dots,n_k)$ resulting from pure strategies:

$$p_{n_1,\dots,n_k} = \mathrm{e}^{-\beta\, H(n_1,\dots,n_k)-\zeta-1}.$$

The Lagrange multiplier ζ can be eliminated by normalization of the weights. This leads to the Gibbs distribution function [8]:

$$p_{n_1,\dots,n_k} = \frac{\mathrm{e}^{-\beta\, H(n_1,\dots,n_k)}}{\sum\limits_{n_1,\dots,n_k=0}^{1} \mathrm{e}^{-\beta\, H(n_1,\dots,n_k)}}\,.$$

Note that we have put no restriction on the properties of $H(n_1,\dots,n_k)$. The complete information about this random variable is contained in the statistical sum

$$Z(\beta) := \sum_{n_1,\dots,n_k=0}^{1} \mathrm{e}^{-\beta\, H(n_1,\dots,n_k)}\,,$$

because its logarithm is the cumulant-generating function (the moments of $H(n_1,\dots,n_k)$ are given by $(-1)^n\,\frac{d^n \ln Z}{d\beta^n}$). Physicists used to call the inverse of the Lagrange multiplier β the temperature T. The expectation of the profit $-E(H(n_1,\dots,n_k))$ is a decreasing function of T and in the limit $T \to 0^+$ it reaches its maximum. It is easy to notice that the statistical sum $Z(\beta)$ factorizes for the profit function given by (1):

$$\sum_{n_1,\dots,n_k=0}^{1} \mathrm{e}^{\beta \sum\limits_{m=1}^{k} h_m n_m} = \sum_{n_1,\dots,n_k=0}^{1} \prod_{m=1}^{k} \mathrm{e}^{\beta\, h_m n_m} = \prod_{m=1}^{k} \sum_{n_m=0}^{1} \mathrm{e}^{\beta\, h_m n_m}\,.$$

This means that the profits made at different moments are independent and there exist a risk-free pure arbitrage strategy of the form: keep the commodity only if the prices are increasing (that is $n_m = \frac{1+\operatorname{sign} h_m}{2}$). The opposing strategy ($n_m = \frac{1-\operatorname{sign} h_m}{2}$) defining a canonical portfolio for $T \to 0^-$ forms a risk-free strategy for short positions. The canonical portfolio representing monkey strategies has infinite temperature, $T=\pm\infty$. To translate the Markowitz portfolio theory into our thermodynamical language we should include in an explicit way the portfolio risk measured by its second cumulant moment $\big(H-E(H)\big)^2$ (and the corresponding Lagrange multiplier!) besides the random variable $H(n_1,\dots,n_k)$. For our aims it suffices to consider only the equilibrium variant of canonical portfolios with average[1] risk given by $\frac{\partial^2 \ln Z}{\partial \beta^2}$.

4 Non-zero transactions costs

If we take transaction costs into consideration[2] then Eq. (1) should be replaced by the more general formula:

$$-\sum_{m=1}^{k} h_m n_m - j\,(n_{m-1}\oplus n_m) \;\;\rightarrow\;\; H(n_1,\dots,n_k)\,, \tag{3}$$

where $n_0 := n_k$ for periodic boundary conditions (otherwise n_0 should be fixed arbitrary). $\oplus$ denotes addition modulo 2, and the constant $j>0$ is equal to the logarithm of the cost of a single transaction. The careful reader will certainly notice that Eq. (3) represent an hamiltonian of an Ising chain [9] (the shift in the sequence (n_m) by $-\frac{1}{2}$ introduces only an unimportant constant to the formula). Now the statistical sum $Z(\beta)$ cannot be factorized in terms of contributions from separate moments. Instead we should use transition matrices that depend on the immediate moments $m-1$ i m. Then for a convenient periodic boundary conditions we arrive at the formula that expresses $Z(\beta)$ as a trace of the product of transition matrices:

$$Z(\beta) = \sum_{n_1,\dots,n_k=0}^{1} M(1)_{n_k\,n_1} M(2)_{n_1\,n_2}\cdots M(k)_{n_{k-1}\,n_k} \;=\; \mathrm{Tr}\prod_{m=1}^{k} M(m)\,,$$

where

$$M(m)_{n_{m-1}\,n_m} := \mathrm{e}^{\beta(h_m\,n_m - j\,(n_{m-1}\oplus n_m))}\,.$$

[1] Markowitz theory deals with effective portfolios that are characterized by minimal risk at a given profit level.

[2] For simplicity we consider only the case of cost constant per unit of the commodity.

Unfortunately, the entries of the matrices $M(m)$ depend on time via h_m and the analysis of the proper value problem does not lead to any compact form of the statistical sum (except for the uninteresting case of the constant sequence (h_m)).

5 The (min,+) algebra of portfolios

The definition of entropy (2) allows to find the following relation among the entropy, average profit and the statistical sum:

$$E(H) + T\,\ln Z = T\,S\,. \tag{4}$$

The entropy is positive and bounded from above ($S \leq k \ln 2$) therefore Eq. (4) can be used to determine the strategies giving maximal profits:

$$H_\pm := \lim_{T\to 0^\pm} E(H) = -\lim_{T\to 0^\pm} T\,\ln Z = \lim_{\beta\to\pm\infty} \log_{\mathrm{e}^{-\beta}} Z\,.$$

Elementary properties of logarithms[3]

$$\log_\varepsilon(\varepsilon^a\,\varepsilon^b) = a+b\,,\quad \lim_{\varepsilon\to 0^+} \log_\varepsilon(\varepsilon^a+\varepsilon^b) = \min(a,b)$$

imply that the full information about the most profitable strategy is given by the product of logarithms of transition matrices

$$\tilde{M}(m)_{n_{m-1}\,n_m} := \log_{\mathrm{e}^{-\beta}} M(m)_{n_{m-1}\,n_m} = -h_m\,n_m + j\,(n_{m-1}\oplus n_m)$$

if we replace addition of real numbers by the operation min of taking the minimal element of them and multiplication of numbers by their sum (that is by using the (min,+) algebra [10])

$$\big(\tilde{M}(m)\times\tilde{M}(m{+}1)\big)_{n_{m-1}\,n_{m+1}} := \min_{n_m}\big(\tilde{M}(m)_{n_{m-1}\,n_m} + \tilde{M}(m{+}1)_{n_m\,n_{m+1}}\big)\,. \tag{5}$$

Investigation of the matrix elements contributing to the "product" allows to reconstruct the sequence $(n_1,\ldots,n_k)$ and its minimal element will correspond to the maximal available profit in the game.

[3] For short positions strategies ($T\to 0^-$) we should find the limit $\lim_{\varepsilon\to\infty} \log_\varepsilon(\varepsilon^a+\varepsilon^b) = \max(a,b)$

6 Arbitrage risk

For a given price $(h_1, \ldots, h_k)$ let us call the potential arbitrage strategy any strategy that if completed with element corresponding to moments $k' > k$ might turn out to be the strategy giving maximal profit for the hamiltonian $H(n_1, \ldots, n_k, \ldots, n_{k'})$. In our case there are only four potential arbitrage strategies if the initial value n_0 is not fixed. In general, in a market with $N-1$ commodities there are $2N$ such strategies. It is easy to notice that for a given price sequence $(h_1, \ldots, h_k)$ potential arbitrage strategy has the form

$$(0, 1, 1, 0, 1, 0, 0, n_{k-l+1}, n_{k-l+2}, \ldots, n_k)\,,$$

and can be decomposed into two parts. The first one constructed according to the knowledge of the sequence $(h_1, \ldots, h_k)$ and the second one of the length l. We will call l the coherence depth. The functional dependence of the coherence depth l on the costs j might form an interesting market indicator of structure of price movements. The final sequence $(n_{k-l+2}, \ldots, n_k)$ can be determined only if prices h_m are known for $m > k$. Therefore any potential arbitrage strategy is an optimal strategy for a player whose profits are known only up to the moment k. In that sense non-vanishing transaction costs involve arbitrage risk that might be caused, for example, by the finite maturity time of contracts or splitting of orders.

The algorithm for finding potential arbitrage strategies, the respective profits and coherence depths is fast because uses only addition of matrices ("product") which is linear in k. But this is insufficient for the risk and profits analysis of all portfolios equivalent to potential arbitrage portfolios. To analyze the arbitrage opportunities we should consider the whole low temperature sector of canonical portfolios $T \in (0, T_+]$, where $E_{T_+}(H) := \max_{n_0, n_k} \big(\tilde{M}(1) \times \cdots \times \tilde{M}(k)\big)_{n_0 n_k}$ because it is the minimal set that contains all potential arbitrage portfolios. Unfortunately due to the lack of compact form of the statistical sum, the knowledge of canonical arbitrage portfolios requires performing of 3^k arithmetical operations what is a difficult computational task.

7 Simulations of canonical portfolios

Simulations are usually perceived as modelling of real processes by a Turing machines. But complexity of various phenomena shows the limits of effective polynomial algorithms. Is seems that the future will reverse the roles: we will compute by simulations perceived as measurements of appropriate (physical?) phenomena. In fact such methods have been used for centuries[4]. The model discussed above can be easily associated with quantum computation (and games). Calculations for portfolios should take into consideration all available pure strategies whose number grows exponentially in k (2^k). Therefore the classical Turing machines are of little use. One of the future possibilities might be exploration of nano-structures having properties of Ising chains. Changes of local magnetic fields h_m and controlling temperature may allow for effective determination of profits and strategies for players of various abilities (measured by their temperature [3, 4]). The values of the parameters n_m would be found by measurements of magnetic moments. Another effective method might consist in using quantum parallelism for simultaneous determination of all 2^k components of the statistical sum. Quantum computation would use superpositions of k qubit quantum states

$$\mathbb{C}P^{2^k-1} \ni |\psi\rangle := \sum_{n_1,\ldots,n_k=0}^{1} \mathrm{e}^{\mathrm{i}\,\varphi_{n_1\ldots n_k}+\frac{\beta}{2}\sum_{m=1}^{k}(h_m n_m - j\,(n_{m-1}\oplus n_m))}|n_1\rangle\otimes\ldots\otimes|n_k\rangle$$

with arbitrary phases $\varphi_{n_1\ldots n_k}$. Measurements of the states $|\psi\rangle$ would allow to identify for a given portfolio all important leading terms in the statistical sum. The paper [11] presents analysis of the problem of simulation of an Ising chain on a quantum computer. One can easily identify the unitary transformations used there with transition matrices for probability amplitudes. Details of such computations and their interpretation in term of quantum market games will be presented in a separate paper (cf [12]).

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