

The next stage: quantum game theory

(RePEc:sla:eakjkl:18 31-VII-2003)

Edward W. Piotrowski

Institute of Theoretical Physics, University of Białystok,
Lipowa 41, Pl 15424 Białystok, Poland
e-mail: ep@alpha.uwb.edu.pl

Jan Śladkowski

Institute of Physics, University of Silesia,
Uniwersytecka 4, Pl 40007 Katowice, Poland
e-mail: sladk@us.edu.pl

Abstract

Recent development in quantum computation and quantum information theory allows to extend the scope of game theory for the quantum world. The paper presents the history, basic ideas and recent development in quantum game theory. On grounds of the discussed material, we reason about possible future development of quantum game theory and its impact on information processing and the emerging information society.

PACS Classification: 02.50.Le, 03.67.Lx, 05.50.+q, 05.30.-d

Mathematics Subject Classification: 81-02, 91-02, 91A40, 81S99

Keywords and phrases: quantum games, quantum strategies, quantum information theory, quantum computations

1 Introduction

The emerging of global information infrastructure caused one of the main paradigm shifts in human history: information is becoming a crucial if not the most important resource. Recently the scientific community has become more and more aware that information processing is a physical phenomenon and that information theory is inseparable from both applied and fundamental physics. Attention to the very physical aspects of information processing revealed new perspectives of computation, cryptography and communication methods. In most of the cases quantum description of the system provides advantages over the classical situation. Game theory, the study of (rational) decision making in conflict situations, seems to ask for a quantum version. For example, games against nature [1] include those for which nature is quantum mechanical. Does quantum theory offer more subtle ways of playing games? Game theory considers strategies that are probabilistic mixtures of pure strategies. Why cannot they be intertwined in a more complicated way, for example interfered or entangled? Are there situations in which quantum theory can enlarge the set of possible strategies? Can quantum strategies be more successful than classical ones? And if the answer is yes are they of any practical value? John von Neumann is one of the founders of both game theory [2] and quantum theory, is that a meaningful coincidence? In this paper we would like to convince the reader that the research on quantum game theory cannot be neglected because present technological development suggest that sooner or later someone would take full advantage of quantum theory and may use quantum strategies to beat us at some game. Cryptography and communication methods seem to be the more probable battle fields but who can be sure? The paper is organized as follows. We will begin by presenting a detailed analysis of a simple example given by David A. Meyer [5] that will illustrate the general idea of a quantum game and methods of gaining an advantage over "classical opponent". Then we will attempt at giving a definition of a quantum game and review problems that have already been discussed in the literature. Finally we will try to show some problems that should be addressed in the near future. In the following discussion we will use quantum theory as a language but the broadcasted message would be that it can be used as a weapon.

2 Quantum game prehistory and history

It is not easy to give the precise date of birth of quantum game theory. Quantum games have probably been camouflaged since the very beginning of the quantum era because a lot of experiments can be reformulated in terms of game theory. Quantum game theory began with works of Wiesner on quantum money [3], Vaidman, who probably first used the term game in quantum context [4], and Meyer [5] and Eisert et al [6] who first formulated their problems in game theory formalism. Possible applications of quantum games in biology were discussed by Iqbal and Toor [7], in economics by Piotrowski and Ślaskowski [8, 9]. Flitney and Abbott quantized Parrondo's paradox [10]. David Meyer put forward a fabulous argument for research on quantum game theory that we are going to retell here [5]. He describes a game that is likely to be played by two characters of the popular TV series *Star Trek: The Next Generation*, Captain Picard and Q. Suppose they play the modern version of the penny flip game that is implemented as a *spin-flip game* (there probably are no coins on a starship). Picard is to set an electron in the spin up state, whereupon they will take turns (Q, then Picard, then Q) flipping the spin or not, without being able to see it. Q wins if the spin is up when they measure the electron's state. This is a two-person *zero-sum strategic game* which might be analyzed using the payoff matrix:

	<i>NN</i>	<i>NF</i>	<i>FN</i>	<i>FF</i>
<i>N</i>	−1	1	1	−1
<i>F</i>	1	−1	−1	1

where the rows and columns are labelled by Picard's and Q's *pure strategies* (moves), respectively; *F* denotes a flip and *N* denotes no flip; and the numbers in the matrix are Picard's payoffs: 1 indicating a win and −1 a loss of a one currency unit. Q's payoffs can be obtained by reversing the signs in the above matrix (this is the defining feature of a zero sum game).

Example: Q's strategy is to flip the spin on his first turn and then not flip it on his second, while Picard's strategy is to not flip the spin on his turn. The result is that the state of the spin is, successively: *U*, *D*, *D*, *D*, so Picard wins.

It is natural to define a two dimensional vector space *V* with basis (*U*, *D*) and to represent players' strategies by sequences of 2×2 matrices. That is,

the matrices

$$F := \begin{array}{c} U \quad D \\ U \quad D \end{array} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad N := \begin{array}{c} U \quad D \\ U \quad D \end{array} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

correspond to flipping and not flipping the spin, respectively, since we define them to act by left multiplication on the vector representing the state of the spin. A general *mixed strategy* consists in a linear combination of F and N , which acts as a 2×2 matrix:

$$\begin{array}{c} U \quad D \\ U \quad D \end{array} \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

if the player flips the spin with probability $p \in [0, 1]$. A sequence of mixed actions puts the state of the electron into a convex linear combination $aU + (1-a)D$, $0 \leq a \leq 1$, which means that if the spin is measured the electron will be in the spin-up state with probability a . Q, having studied quantum theory, is utilizing a *quantum strategy*, implemented as a sequence of unitary, rather than stochastic, matrices. In standard Dirac notation the basis of V is written $(|U\rangle, |D\rangle)$. A *pure* quantum state for the electron is a linear combination $a|U\rangle + b|D\rangle$, $a, b \in \mathbb{C}$, $a\bar{a} + b\bar{b} = 1$, which means that if the spin is measured, the electron will be in the spin-up state with probability $a\bar{a}$. Since the electron starts in the state $|U\rangle$, this is the state of the electron if Q's first action is the unitary operation

$$U_1 = U(a, b) := \begin{array}{c} U \quad D \\ U \quad D \end{array} \begin{pmatrix} a & b \\ \bar{b} & -\bar{a} \end{pmatrix}.$$

Captain Picard is utilizing a *classical mixed strategy* (probabilistic) in which he flips the spin with probability p (has he preferred drill to studying quantum theory?). After his action the electron is in a mixed quantum state, i.e., it is in the pure state $b|U\rangle + a|D\rangle$ with probability p and in the pure state $a|U\rangle + b|D\rangle$ with probability $1-p$. Mixed states are conveniently represented as *density matrices*, elements of $V \otimes V^\dagger$ with trace 1; the diagonal entry (k, k) is the probability that the system is observed to be in the state $|\psi_k\rangle$. The density matrix for a pure state $|\psi\rangle \in V$ is the projection matrix

$|\psi\rangle\langle\psi|$ and the density matrix for a mixed state is the corresponding convex linear combination of pure density matrices. Unitary transformations act on density matrices by conjugation: the electron starts in the pure state $\rho_0 = |U\rangle\langle U|$ and Q's first action puts it into the pure state:

$$\rho_1 = U_1 \rho_0 U_1^\dagger = \begin{pmatrix} a\bar{a} & a\bar{b} \\ b\bar{a} & b\bar{b} \end{pmatrix}.$$

Picard's mixed action acts on this density matrix, not as a stochastic matrix on a probabilistic state, but as a convex linear combination of unitary (deterministic) transformations:

$$\rho_2 = p F \rho_1 F^\dagger + (1-p) N \rho_1 N^\dagger = \begin{pmatrix} p b \bar{b} + (1-p) a \bar{a} & p b \bar{a} + (1-p) a \bar{b} \\ p a \bar{b} + (1-p) b \bar{a} & p a \bar{a} + (1-p) b \bar{b} \end{pmatrix}.$$

For $p = \frac{1}{2}$ the diagonal elements of ρ_2 are equal to $\frac{1}{2}$. If the game were to end here, Picard's strategy would ensure him the expected payoff of 0, independently of Q's strategy. In fact, if Q were to employ any strategy for which $a\bar{a} \neq b\bar{b}$, Picard could obtain the expected payoff of $|a\bar{a} - b\bar{b}| > 0$ by setting $p = 0, 1$ according to whether $b\bar{b} > a\bar{a}$, or the reverse. Similarly, if Picard were to choose $p \neq \frac{1}{2}$, Q could obtain the expected payoff of $|2p - 1|$ by setting $a = 1$ or $b = 1$ according to whether $p < \frac{1}{2}$, or the reverse. Thus the mixed/quantum equilibria for the two-move game are pairs $([\frac{1}{2}F + \frac{1}{2}N], [U(a, b)])$ for which $a\bar{a} = b\bar{b} = \frac{1}{2}$ and the outcome is the same as if both players utilize optimal mixed strategies. But Q has another move at his disposal (U_3) which again transforms the state of the electron by conjugation to $\rho_3 = U_3 \rho_2 U_3^\dagger$. If Q's strategy consists of $U_1 = U(1/\sqrt{2}, 1/\sqrt{2}) = U_3$, his first action puts the electron into a simultaneous eigenstate of both F and N (eigenvalue 1), which is therefore invariant under any mixed strategy $pF + (1-p)N$ of Picard. His second move inverts his first move and produces $\rho_3 = |U\rangle\langle U|$. That is, with probability 1 the electron spin is up! Since Q can do no better than to win with probability 1, this is an optimal quantum strategy for him. All the pairs

$$([pF + (1-p)N], [U(1/\sqrt{2}, 1/\sqrt{2}), U(1/\sqrt{2}, 1/\sqrt{2})])$$

are mixed/quantum equilibria, with value -1 to Picard; this is why he loses every game. We think that this hypothetical story convinces the reader that

quantum games should be studied thoroughly in order to prevent analogous events from shaping his/her destiny let alone other aspects. The practical lesson that the above example teaches is that quantum theory may offer strategies that at least in some cases give advantage over classical strategies. Therefore physicist and game theorists should find answers to the following five questions.

- Is the idea of quantum game feasible?
- Under what conditions some players may be able to take the advantage of quantum phenomena?
- Are there genuine quantum games that have no classical counterparts or origin?
- Are protocols for playing quantum game against human player secure against cheating?
- Can the formalism be generalized to include other non-Boolean logic based systems?

Finding answers to the above questions is challenging and intriguing. It is anticipated that answers to these questions will have a profound impact on the development of quantum theory, quantum information processing and technology. Unfortunately, at this stage it we are not able to give any definite answer and, in fact, we have no idea in what direction we should look to find them. Nevertheless one can present some strong arguments for developing quantum theory of games. Modern technologies are developed mostly due to investigation into the quantum nature of matter. The results of recent experiments in nanotechnology, quantum dots and molecular physics are very promising. This means that we sooner or later may face situations analogous to captain Picard's if we are not on alert. Many cryptographic and information processing problems can be reformulated in game-like setting. Therefore quantum information and quantum cryptography should provide us with cases in point. It is obvious that some classical games can be implemented in such a way that the set of possible strategies would include strategies that certainly deserve the the adjective quantum [11, 12]. Such games can certainly be played in a laboratory. This process is often referred to as quantization of the respective standard game. But this is an abuse of language: we are in fact defining a new game. In

the classical setting the problems of security and honesty are usually well defined. Realistic quantum cryptography systems and quantum networks (BBN, Harvard and Boston Universities are already building the DARPA Quantum Network [13]) will certainly provide us with examples of genuine quantum games and strategies. Unlike, in quantum game theory the problem is much more involved. In many cases it can be settled in the "classical way" (e.g. by selecting arbiters or sort of clearinghouses) but if you admit quantum strategies in less definite setting of actually being developed technologies it may be even difficult to name dishonesty. If quantum games should ever be applied outside physical laboratories a lot of technical problems must be solved. Security of quantum games is only one of them but it already involves error corrections, quantum state tomography and methods of communications and preparations of quantum systems forming the "quantum board" and the necessary "quantum memory". We envisage that critical analysis of already proposed quantum information processing protocols must be done to this end. One of the main objectives would be a definition of (possibly universal) primitives necessary for realistic quantum games. Quantum phenomena probably play important role in biological and other complex systems and, although this point of view is not commonly accepted, quantum games may turn out to be an important tool for the analysis of various complex systems. Genetic algorithms and DNA computation can also be used to implement games and quantum games may be the most promising field [14]. Massive parallel DNA processing would allow to play simultaneously trillions of games. Noncommutative propositions are characteristic of various situations not necessary associated with quantum systems. In fact, the richness of possible structures is immense. There are suggestions that quantum-like description of market phenomena may be more accurate than the classical (probabilistic) one [15]. The quantum morphogenesis [16] shows one possible way of generalization of the formalism that may find application in social sciences.

3 Quantum game theory

Basically, any quantum system that can be manipulated by at least one party and where the utility of the moves can be reasonably defined, quantified and ordered may be conceived as a quantum game. The quantum system may be referred to as a *quantum board* although the term *universum*

of the game seems to be more appropriate [17]. We will suppose that all players know the state of the game at the beginning and at some crucial stages that may depend on the game being played. This is a subtle point because it is not always possible to identify the state of a quantum system let alone the technical problems with actual identification of the state (one can easily give examples of systems that are only partially accessible to some players [18]). A "realistic" quantum game should include measuring apparatuses or information channels that provide information on the state of the game at crucial stages and specify the way of its termination. We will neglect these nontrivial issues here. Therefore we will suppose that a *two-player quantum game* $\Gamma = (\mathcal{H}, \rho, S_A, S_B, P_A, P_B)$ is completely specified by the underlying Hilbert space $\mathcal{H}$ of the physical system, the initial state $\rho \in \mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H})$ is the associated state space, the sets S_A and S_B of permissible quantum operations of the two players, and the pay-off (utility) functions P_A and P_B , which specify the pay-off for each player. A *quantum strategy* $s_A \in S_A, s_B \in S_B$ is a collection of admissible quantum operations, that is the mappings of the space of states onto itself. One usually supposes that they are completely positive trace-preserving maps. The quantum game's definition may also include certain additional rules, such as the order of the implementation of the respective quantum strategies or restriction on the admissible communication channels, methods of stopping the game etc. We also exclude the alteration of the pay-off during the game. The generalization for the N players case is obvious. Schematically we have:

$$\rho \mapsto (s_A, s_B) \mapsto \sigma \Rightarrow (P_A, P_B).$$

The following concepts will be used in the remainder of this paper. These definitions are completely analogous to the corresponding definitions in standard game theory [19, 20]. The adjective quantum gives no extra meaning to them. A strategy s_A is called a *dominant strategy* of Alice if

$$P_A(s_A, s'_B) \geq P_A(s'_A, s'_B)$$

for all $s'_A \in S_A, s'_B \in S_B$. Analogously we can define a dominant strategy for Bob. A pair (s_A, s_B) is said to be an *equilibrium in dominant strategies* if s_A and s_B are the players' respective dominant strategies. A combination of strategies (s_A, s_B) is called a *Nash equilibrium* if

$$\begin{aligned} P_A(s_A, s_B) &\geq P_A(s'_A, s_B), \\ P_B(s_A, s_B) &\geq P_B(s_A, s'_B). \end{aligned}$$

A pair of strategies (s_A, s_B) is called *Pareto optimal*, if it is not possible to increase one player's pay-off without lowering the pay-off of the other player. A solution in dominant strategies is the strongest solution concept for a non-zero sum game. For example, in the popular Prisoner's Dilemma game [19, 20]:

	Bob : C	Bob : D
Alice : C	(3, 3)	(0, 5)
Alice : D	(5, 0)	(1, 1)

where the numbers in parentheses represent the row (Alice) and column (Bob) player's payoffs, respectively. Defection (D) is the dominant strategy, as it is favorable regardless what strategy the other party chooses.

In general the optimal strategy depends on the strategy chosen by the other party. A Nash equilibrium implies that neither player has a motivation to unilaterally alter his/her strategy from this kind of equilibrium solution, as this action will lower his/her pay-off. Given that the other player will stick to the strategy corresponding to the equilibrium, the best result is achieved by also playing the equilibrium solution. The concept of Nash equilibria is therefore of paramount importance to studies of non-zero-sum games. It is, however, only an acceptable solution concept if the Nash equilibrium is not unique (this happens very often). For games with multiple Nash equilibria we have to find a way to eliminate all but one of them. Therefore a Nash equilibrium is not necessarily an efficient and satisfactory one. We say that an equilibrium is Pareto optimal if there is no other outcome which would make both players better off. But usually there are no incentives to adopt the Pareto optimal strategies. In the Prisoner's Dilemma the Pareto equilibrium is reached if both players adopt the strategy C, but they are afraid of being outwitted by the opponent's playing D. If the same game is repeated many times the situation changes because the players may communicate by changing strategies and learning is possible. To this end both players should adopt mixed strategies. One can prove that any game has a Nash equilibrium in the class of mixed strategies [19, 20].

4 Quantum games in action: a review recent results

Quantum game theory attracted much attention since the prescription for quantization of games has been put forward by Eisert, Wilkens and Lewenstein [6]. It was subsequently generalized by Marinatto and Weber [21]. This general setting was described above. Actually, it can be applied to any $2 \times n$ games (each player has n strategies and the players' actions are represented by $U(n)$ or $SU(n)$ operators). This prescription has been used for "quantizing" various classical games (Prisoner's Dilemma [6, 22], The Monty Hall Problem [23, 24], Battle of Sexes [25, 26, 27], Stag Hunt Game [28], Rock, Scissors and Paper [29, 30], Coordination Problem [31], Duopoly Problem [32], duels [33] etc. The results show that, in general, the "quantization process" and relations to the background classical problems are not unique. Nash equilibria can be found but, as in the classical problems, in most cases they are not Pareto optimal. Lee and Johnson have shown that playing games quantum mechanically can be more efficient and giving a saturation of the upper bound on the efficiency [34]. From their work it can be deduced that there are quantum versions of the minimax theorem for zero sum quantum games and the Nash equilibrium theorem for general static quantum games. There are many unexplored connections between quantum information theory and other scientific models. Quantum game theory offers tools in analysis of phenomena that usually are not considered as physical processes. Theory of information can be used for analysis of algorithms that describe player's strategies and tactics but classical games form only a small? subclass of games that can be played in quantum information media. If we ignore technological problem then we can extend this subclass so that exploration of quantum phenomena is possible. There are two obvious modifications of classical simulation games.

1 – prequantization: Redefine the game so that it became a reversal operation on qubits representing player's strategies. This allows for quantum coherence of strategies¹.

2 – quantization: Reduce the number of qubits and allow arbitrary unitary² transformation so that the basic feature of the classical game are

¹This may result from nonclassical strategies or classically forbidden measurements of the state of the game

preserved.

Any game modified in this way is in fact a quantum algorithm that usually allows for more effective information processing than the starting game. Actually, any quantum computation is a potential quantum game if we manage to reinterpret it in game-theoretical terms. To illustrate the second method let us consider Wiesner's counterfeit-proof banknote [3]. This is the first quantum secrecy method (elimination of effective eavesdropping). As a quantum game it consists in a finite series of sub-games presented in Fig. 1. An arbiter Trent produces a pair of random qubits $|\psi_T\rangle$ and $|\psi_{T'}\rangle$. The polarization of the qubit (strategy) $|\psi_T\rangle$ is known to Trent and is kept secret. The qubit $|\psi_{T'}\rangle$ is ancillary. Alice qubit $|\psi_A\rangle$ describes her strategies $|I\rangle$ and $|0\rangle$. The first move is performed by Alice. Her strategy $|I\rangle$ consists in switching the Trent's qubits $|\psi_T\rangle$ and $|\psi_{T'}\rangle$. The strategy $|0\rangle$ consists in leaving the Trent's qubits intact. These moves form the controlled-swap gate [35]. Her opponent Bob wins only if after the game Trent learns that his qubit $|\psi_T\rangle$ has not been changed.

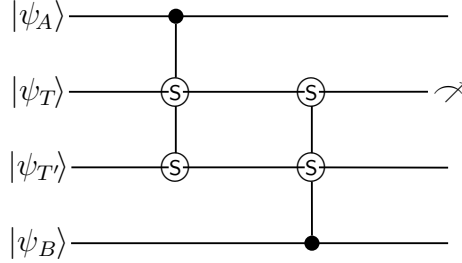


Figure 1: Quantum identification game constructed from two controlled-swap gates (Wiesner's money).

To win Bob must always begin with a strategy identical to the one used by Alice. If there is no coordination of moves between Alice and Bob the probability of Bob's success exponentially decreases with growing number of sub-games being played and is negligible even for a small number of sub-games. Alice and Bob's strategies are classical but due to the pre-quantization process eavesdropping is not possible if Trent uses arbitrary polarizations $|\psi_T\rangle = |0\rangle + z|I\rangle$, $z \in \overline{\mathcal{C}} \simeq S_2$ (in the projective nonhomogeneous coordinates). This game can be quantized by elimination of the

²At least one of the performed operations should not be equivalent to a classical one

ancillary qubit $|\psi_T\rangle$. Then Alice and Bob strategies should be equivalent to controlled-Hadamard gates [35]. In this case Trent's qubit is changed only if Alice adopts the strategy $|I\rangle$ that result in $|\psi_T\rangle = |0\rangle + z|I\rangle \longrightarrow |0\rangle + \frac{1-z}{1+z}|I\rangle$ (quantum Fourier transform), see Fig. 2. The actual Wiesner's idea was to encode the secret values of $|\psi_T\rangle$ that result from Alice moves in the series of sub-games on an otherwise numbered banknote. In addition, the issuer Trent takes over the role of Alice and records the values of $|\psi_T\rangle$ and $|\psi_A\rangle$ with the label being the number of the banknote. The authentication of the banknote is equivalent to a success in the game when Bob's strategy is used against that recorded by Trent (if Bob wins then his forgery is successful).

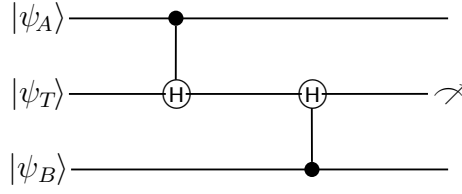


Figure 2: Quantum identification game constructed from two controlled-Hadamard gates.

The introduction of classically impossible strategies results in better security against quantum attack (pretending to be Alice). Eavesdropping of the state $|\psi_T\rangle$ modified by Alice's strategy is ineffective even if Trent limits himself to polarizations from the set $\{|0\rangle, |I\rangle\}$. It is possible that an analogous reduction of qubits allows to exponentially reduce the complexity of quantum algorithms. Therefore quantum games may sometimes be the only feasible alternatives if the classical problems are computationally too complex to be ever implemented. Description of games against Nature is far more complicated. It is not easy to show that they do not contradict known natural laws or are actually being played. For example, let us consider a prequantized version of the Maxwell's Demon game against Nature [36]. Demon acting in accord with physical laws tries to build a Szilard's engine. The Demon fails because Nature erases information and this is an energy consuming process. Such arguments work also in biology and social sciences. The classical theory of interacting particles localized at nodes of crystal lattices results in quantum model of collective phenomenon known as phonon. Phonons do not exist outside the crystal lattice. May humans

and animals form classical ingredients of large quantum entities? May Penrose will not be able to find consciousness at the sub-neuronic microtubular level [37] because it is localized at the complex multi-neuron level? Do we try to convince ourselves that there are living actors in TV sets?

4.1 Quantum games in economics and social sciences

Modern game theory has its roots in economics and social sciences and one should not be surprised by number of attempts at quantizing classical problems. In the "standard" quantum game theory one tries in some sense to quantize an operational description of "classical" versions of the game being analyzed. It usually enlarges the set admissible strategies in a nontrivial way. Piotrowski and Śladkowski follow a different way. Technological development will sooner or later result in construction of quantum computers. If one considers the number of active traders, the intensity of trade contemporary markets and their role in the civilization then one must admit that the *the market game* is the biggest one ever played by humans. How would look a market cleared by a quantum computer quantum network or other quantum device? They propose to describe market players strategies in terms of state vectors $|\psi\rangle$ belonging to some Hilbert space $\mathcal{H}$ [8, 9]. The probability densities of revealing the players, say Alice and Bob, intentions are described in terms of random variables p and q :

$$\frac{|\langle q|\psi\rangle_A|^2}{{}_A\langle\psi|\psi\rangle_A} \frac{|\langle p|\psi\rangle_B|^2}{{}_B\langle\psi|\psi\rangle_B} dqdp,$$

where $\langle q|\psi\rangle_A$ is the probability amplitude of offering the price q by Alice who wants to buy and the demand component of her state is given by $|\psi\rangle_A \in \mathcal{H}_A$. Bob's amplitude $\langle p|\psi\rangle_B$ is interpreted in an analogous way (opposite position). Of course, the "intentions" q and p not always result in the accomplishment of the transaction [8]. According to standard risk theory it seems reasonable to define the observable of *the risk inclination operator*:

$$H(\mathcal{P}_k, \mathcal{Q}_k) := \frac{(\mathcal{P}_k - p_{k0})^2}{2m} + \frac{m\omega^2(\mathcal{Q}_k - q_{k0})^2}{2},$$

where $p_{k0} := \frac{{}_k\langle\psi|\mathcal{P}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $q_{k0} := \frac{{}_k\langle\psi|\mathcal{Q}_k|\psi\rangle_k}{{}_k\langle\psi|\psi\rangle_k}$, $\omega := \frac{2\pi}{\theta}$. θ denotes the characteristic time of transaction [38] which is, roughly speaking, an average time spread between two opposite moves of a player (e.g. buying and selling the

same asset). The parameter $m > 0$ measures the risk asymmetry between buying and selling positions. Analogies with quantum harmonic oscillator allow for the following characterization of quantum market games. The constant $\hbar_E$ describes the minimal inclination of the player to risk. It is equal to the product of the lowest eigenvalue of $H(\mathcal{P}_k, \mathcal{Q}_k)$ and 2θ . 2θ is in fact the minimal interval during which it makes sense to measure the profit. Except the ground state all the adiabatic strategies $H(\mathcal{P}_k, \mathcal{Q}_k)|\psi\rangle = \text{const}|\psi\rangle$ are giffens [8, 39] that is goods that do not obey the law of demand and supply. It should be noted here that in a general case the operators $\mathcal{Q}_k$ do not commute because traders observe moves of other players and often act accordingly. One big bid can influence the market at least in a limited time spread. Therefore it is natural to apply the formalism of noncommutative quantum mechanics where one considers

$$[x^j, x^k] = \mathbf{i} \Theta^{jk} := \mathbf{i} \Theta \epsilon^{jk}.$$

The analysis of harmonic oscillator in more than one dimensions imply that the parameter Θ modifies the constant $\hbar_E \rightarrow \sqrt{\hbar_E^2 + \Theta^2}$ and, accordingly, the eigenvalues of $H(\mathcal{P}_k, \mathcal{Q}_k)$. This has the natural interpretation that moves performed by other players can diminish or increase one's inclination to taking risk. When a game allows a great number of players in it is useful to consider it as a two-players game: the trader $|\psi\rangle_k$ against the Rest of the World (RW). The concrete algorithm $\mathcal{A}$ may allow for an effective strategy of RW (for a sufficiently large number of players a single player would not have much influence on the form of the RW strategy). If one considers the RW strategy it make sense to declare its simultaneous demand and supply states because for one player RW is a buyer and for another it is a seller. To describe such situation it is convenient to use the Wigner formalism. If the market continuously measures the same strategy of the player, say the demand $\langle q|\psi\rangle$, and the process is repeated sufficiently often for the whole market, then the prices given by some algorithm do not result from the supplying strategy $\langle p|\psi\rangle$ of the player. The necessary condition for determining the profit of the game is the transition of the player to the state $\langle p|\psi\rangle$. If, simultaneously, many of the players change their strategies then the quotation process may collapse due to the lack of opposite moves. In this way the quantum Zeno effects explains stock exchange crashes. Another example of the quantum market Zeno effect is the stabilization of prices of an asset provided by a monopolist.

But one does not need any sophisticated equipment or technology to apply quantum theory and quantum games in economics and social sciences. In fact, Lambertini claims that quantum mechanics and mathematical economics are isomorphic [40]. Therefore one should expect that various quantum tools as the quantum morphogenesis [16] would be invented and used to describe social phenomena. An interesting analysis was done by Arfi who proposes to use quantum game for wide spectrum of problems in political sciences [41]. Quantum game theory may help solving some philosophical paradoxes, c.f. the quantum solution to the notorious Newcomb's paradox (free will dilemma) [42].

Strategies adopted by social groups or their individual members usually seem to be unspeakable and elusive. Efforts to imitate them often fail. Impossible to clone quantum strategies have analogous properties [35]. On the other side, customs, habits and memories are so durable that possibilities of effacing them are illusory. This resembles the no deleting theorem for quantum states (strategies) [43]. Both theorems describe the same forbidden process expressed in reverse chronological order $|\psi\rangle|0\rangle \leftrightarrow |\psi\rangle|\psi\rangle$. An interesting thermodynamical discussion of this impossibility is given in [44]. These analogies cannot be explained in the classical paradigm.

4.2 Quantum games in biology

Living organism may in fact behave in quantum-like way. This may be caused by at least two factors. First, quantum entanglement and decoherence may affect various molecular processes. Second, quantum-like description of dynamics in a population of interacting individuals may be more accurate than the probabilistic one. Therefore a cautious speculation on the possibility that the natural world might already be exploiting the advantages of quantum games on the macroscopic scale may be in place. Maynard Smith in his book *Evolution and the Theory of Games* [45] discusses an evolutionary approach in classical game theory. The concept of evolutionary stability stimulated the development of evolutionary game theory. Iqbal and Toor showed in a series of papers [7, 29, 46, 47, 48] that the presence of entanglement, in asymmetric as well as symmetric bimatrix games, can disturb the evolutionary stability expressed by the idea of evolutionary stable strategies. Therefore evolutionary stability of a symmetric Nash equilibrium can be made to appear or disappear by controlling en-

tanglement in symmetric and asymmetric bimatrix games. It shows that the presence of quantum mechanical effects may have a deciding role on the outcomes of evolutionary dynamics in a population of interacting entities. They suggest that a relevance of their ideas may be found in the studies of the evolution of genetic code at the dawn of life and evolutionary algorithms where interactions between individual of a population may be governed by quantum effects. The nature of these quantum effects, influencing the course of evolution, will also determine the evolutionary outcome. Therefore Darwin's idea of natural selection may be relevant even for quantum systems. Another class of problem concerns replication in biology [49]. The role of DNA and its replication still waits for explanation. Game theory and quantum game theory offer interesting and powerful tools to this end the results will probably find their applications in computation, physics, complex system analysis and cognition sciences [50, 51, 52]. Neuroeconomics [53, 54], a discipline that aims at proving that economic theory may provide an alternative to the classical Cartesian model of the brain and behavior, is a source of fascinating topics for a debate. Is there a *quantum neuroeconomics* [55]? We may expect a rich dialogue between theoretical neurobiology and quantum logic [56].

4.3 Quantum games and information processing

Quantum theory of information is certainly a serious challenge to the standard game theory and will probably stimulate the research in quantum game theory. Most of the cryptographic problems are in fact games, sometimes in camouflage. Analysis and design of cryptographic primitives can also be perceived as games so their quantum counterparts are quantum games, e.g. quantum key distribution, quantum coin tossing [57, 13] or the coordination problem in distributive computing [58]. Coin tossing protocols form an important class of cryptographic primitives. They are used to define a random bit among separated parties. Classical coin flipping can implemented by a trusted arbiter or by assuming that the players have limited computational power. If the players have unlimited computational power then no classical coin flipping protocol is possible because any such protocol represents a two player game and, according to game theory, there always is a player with a winning strategy. By contrast, in a quantum world the existence of coin flipping protocols is not ruled out even by unlimited computational power. This is because any attempt by a

player to deviate (cheat) from the protocol can disturb the quantum states, and therefore be detected by the adversary. But this is far from being the whole story. There are quantum games that live across the border of our present knowledge. For example, consider some classical or quantum problem X . Let us define the game $kXcl$: you win if and only if you solve the problem (perform the task) X given access to only k bits of information. The quantum counterpart reads: solve the problem X on a quantum computer or other quantum device given access to only k bits of information. Let us call the game $kXcl$ or kXq interesting if the corresponding limited information-tasks are feasible. Let $OkhamXcl$ ($OkhamXq$) denotes the minimal k interesting game in the class $kXcl$ (kXq). Authors of the paper [55] described the game played by a market trader who gains the profit P for each bit (qubit) of information about her strategy. If we denote this game by MP then $OkhamM\frac{1}{2}cl = 2M\frac{1}{2}cl$ and for $P > \frac{1}{2}$ the game does not exist $OkhamMPcl$. They also considered the more effective game $1M\frac{2+\sqrt{2}}{4}q$ for which $OkhamM\frac{2+\sqrt{2}}{4}q \neq 1M\frac{2+\sqrt{2}}{4}q$ if the trader can operate on more than one market. This happens because there are entangled strategies that are more profitable [59]. There are a lot of intriguing questions that can be ask, for example for which X the meta-game $Okham(OkhamXq)cl$ can be solved or when, if at all, the meta-problem $Okham(OkhamXq)q$ is well defined problem. Such problems arise in quantum memory analysis [60]. Algorithmic combinatorial games, except for cellular automata, have been completely ignored by quantum physicists. This is astonishing because at least some of the important intractable problems might be attacked and solved on a quantum computer (even such a simple one player game as Minesweeper in NP-complete [61]).

4.4 Quantum games, complexity theory and decision theory

What form does the decision theory take for a quantum player? Almost all quantum acts involve preparing a system, measuring it, and then receiving some reward (in a more or less general sense) which is dependent on the outcome of the measurement. Therefore it should not be astonishing that game-theoretical analysis of quantum phenomena has far reaching consequences. Deutsch claims to have derived the Born from decision-theoretic assumptions [62, 63]. In fact he have defined a quantum game and quantum-

mechanical version of decision theory. What is striking about the Deutsch game is that rational agents are so strongly constrained in their behavior that not only must they assign probabilities to uncertain events, they must assign precisely those probabilities given by the Born rule. His proof must be understood in the explicit context of the Everett interpretation, and that in this context it is acceptable [63].

Roughly speaking, one of the main goals of complexity theory is to present lower bounds on various resources needed to solve a certain computational problem. From a cryptographic viewpoint, the most demanding problem is to prove nontrivial lower bounds on the complexity of breaking concrete cryptographic systems. Query complexity on the other side, is an abstract scenario which can be thought of as a game. The goal is to determine some information by asking as few questions as possible see e.g. quantum oracle and their interrogations [64]. A weak form of quantum interactive proof systems known as quantum Merlin–Arthur games [65, 66, 67] defines a whole class of quantum games with wide application in quantum complexity theory and cryptography. Here, powerful Merlin presents a proof and Arthur, who is the verifier, verifies its correctness. The task is to prove a statement without yielding anything beyond its validity (*zero knowledge proofs*).

There are games in which the agents’ strategies do not have adequate descriptions in terms of some Boolean algebra of logic and theory of probability. They can be analyzed according to the rules of quantum theory and the result are promising, see e.g. the Wise Alice game proposed in [68, 69]. This game is a simplified version of the quantum bargaining game [70] restricted to the “quantum board” of the form $[buy, sell] \times [bid, accept]$. Quantum semantic games also belong to this class [71].

4.5 Quantum gambling

At the present stage of our technological development it already is feasible to open *quantum casinos*, where gambling at quantum games would be possible. Of course, such an enterprise would be costly but if you recall the amount of money spent on advertising various products it seems to us that it is a worthy cause. Goldenberg, Vaidman and Wiesner described the following game based on the coin tossing protocol [72]. Alice has two boxes, A and B , which can store a particle. The quantum states of the particle in the boxes are denoted by $|a\rangle$ and $|b\rangle$, respectively. Alice prepares the

particle in some state and sends box B to Bob.
 Bob wins in one of the two cases:

1. If he finds the particle in box B , then Alice pays him 1 monetary unit (after checking that box A is empty).
2. If he asks Alice to send him box A for verification and he finds that she initially prepared a state different from $|\psi_0\rangle = 1/\sqrt{2}(|a\rangle + |b\rangle)$, then Alice pays him R monetary units.

In any other case Alice wins, and Bob pays her 1 monetary unit. They have analyzed the security of the scheme, possible methods of cheating and calculated the average gain of each party as a result of her/his specific strategy. The analysis shows that the protocol allows two remote parties to play a gambling game, such that in a certain limit it becomes a fair game. No unconditionally secure classical method is known to accomplish this task. This game was implemented by Yong-Sheng Zhang et al, [73]. Other proposals based on properties of non-orthogonal states were put forward by Hwang, Ahn, and Hwang [74] and Hwang and Matsumoto [75]. Witte proposed a quantum version of the Heads or Tails game [76]. Piotrowski and Śładkowski suggested that although sophisticated technologies to put a quantum market in motion are not yet available, simulation of quantum markets and auctions can be performed in an analogous way to precision physical measurements during which classical apparatuses are used to explore quantum phenomena. People seeking after excitement would certainly not miss the opportunity to perfect their skills at using "quantum strategies". To this end an automatic game "Quantum Market" will be sufficient and such a device can be built up due to the recent advances in technology [55]. Segre published an interesting detailed analysis of quantum casinos and a Mathematica packages for simulating quantum gambling [77]. His and others considerations show that quantum gambling is closely related to quantum logic, decision theory and can be used for defining a Bayesian theory of quantum probability [78].

5 Conclusions

Games, used both for entertainment and scientific aims, are traditionally modelled as mathematical objects, and therefore are traditionally seen as

mathematical disciplines. However, all processes in the real world are physical phenomena and as such involve noise, various uncertainty factors and, what concerned us here, quantum phenomena. This fact can be used both for the benefit and detriment. Works of Deutsch, Penrose and others seem to be harbingers of the dawn of a quantum game era when consistent quantum information description would be used not only in physics and natural sciences but also in social sciences and economics. The heterogeneity and fruitfulness of quantum computations will certainly stimulate such interdisciplinary studies and technological development. Quantum games broaden our horizons and offer new opportunities for the technology and economics. If human decisions can be traced to microscopic quantum events one would expect that nature would have taken advantage of quantum computation in evolving complex brains. In that sense one could indeed say that quantum computers are playing according to quantum rules. David Deutsch has proposed an interesting unification of theories of information, evolution and quanta [79]. Lambertini put forward arguments for observing Schroedinger cat like objects on real markets [40]. But why quantum social sciences should emerge just now [80]? They could have not emerged earlier because a tournament quantum computer versus classical one is not possible without technological development necessary for a construction of quantum computers. Now it seems to be feasible. Quantum-like approach to market description might turn out to be an important theoretical tool for investigation of computability problems in economics or game theory even if never implemented in real market [81, 15]. It is tempting to "quantize" Karl Popper's ideas [82] expressed in terms of language-games [83]. Such a revision would determine regions of quantum falsification of scientific theories (q-falsification). Should theories that have high falsification and low q-falsification be regarded as restraining development? If this is the case then *the quantum information processing paradigm* (see e.g. [84]) should replace the alternative platonism or mysticism [37]. Of course, as any other disciplines, quantum game theory also has its negative sides but there is no doubt that it will be a crucial discipline for the emerging information society.

Acknowledgements We are greatly indebted to prof. Zbigniew Hasiewicz for critical reading of the manuscript and helpful remarks.

References

- [1] Milnor, J., Games against nature, in Thrall, R. M., Coombs, C. H., Davis, R. L., (eds.), *Decision Processes*, John Wiley & Sons, New York (1954) p. 49.
- [2] von Neumann, J. and Morgenstern, O., *Theory of Games and Economic Behavior*, Princeton University Press, Princeton (1953).
- [3] Wiesner, S., Conjugate coding, *SIGACT News* **15/1** (1983) 78.
<http://kh.bu.edu/qcl/pdf/wiesners198316024137.pdf>.
- [4] Vaidman, L., Variations on the theme of the Greenberger-Horne-Zeilinger proof, *Foundation of Physics* **29** (1999) 615.
- [5] Meyer, D. A., Quantum strategies, *Physical Review Letters* **82** (1999) 1052.
- [6] Eisert, J., Wilkens, M., Lewenstein, M., Quantum Games and Quantum Strategies, *Physical Review Letters* **83** (1999) 3077.
- [7] Iqbal A., Toor A. H., Evolutionarily stable strategies in quantum games, *Physics Letters A* **280** (2001) 249.
- [8] Piotrowski, E. W., Śladrkowski, J., Quantum Market Games, *Physica A* **312** (2002) 208.
- [9] Piotrowski, E. W., Śladrkowski, J., Quantum English Auctions, *Physica A* **318** (2003) 505.
- [10] Flitney A. P., Abbott D., Quantum models of Parrondo's games, *Physica A* **324** (2003) 152.
- [11] Du, J. et al, Experimental realization of quantum games on a quantum computer *Physical Review Letters* **88** (2002) 137902.
- [12] Pietarinen, A., Quantum logic and quantum theory in a game-theoretic perspective, *Open Systems and Information Dynamics* **9** (2002) 273.
- [13] Elliott, C., Pearson, D., Troxel G., Quantum Cryptography in Practice, preprint of SIGCOMM 2003 paper; quant-ph/0307049.

- [14] Wood, D., Bi, H., Kimbrough, S., Wu, D. J., Chen, J., DNA Starts to Learn Poker, in: Jonoska, N., Seeman, N. C., (eds.), *DNA Computing*, Lecture Notes in Computer Science, Springer-Verlag, Berlin (2002).
- [15] Waite, S., *Quantum investing*, Texere Publishing, London (2002).
- [16] Aerts, D., et al, Quantum morphogenesis: A variation on Thom's catastrophe theory, *Physical Review E* **67** (2003) 051926.
- [17] Bugajski, S., Klamka, J., Quantum iterative system, quantum games, and quantum Parrondo effect, in preparation.
- [18] Beltrametti, E. G., Bugajski, S., A classical extension of quantum mechanics, *Journal of Physics A* **28** (1995) 3329.
- [19] Osborne, M. J., *A Course in Game Theory*, MIT Press, Boston (1994).
- [20] Straffin, P. D., *Game Theory and Strategy*, AMS, Rhode Island (1993).
- [21] Marinatto, L., Weber, T., A Quantum Approach To Static Games Of Complete Information, *Physics Letters A* **272** (2000) 291.
- [22] Du, J. et al, Playing Prisoner's Dilemma with Quantum Rules *Fluctuation and Noise Letters* **2** R189 (2002).
- [23] Flitney, A. P., Abbott, D., Quantum version of the Monty Hall problem, *Physical Review A* **65** (2002) 062318.
- [24] D'Ariano, G. M. et al, The Quantum Monty Hall Problem, *Quantum Information and Computing* **2** (2002) 355.
- [25] Nawaz, A., Toor A. H., Worst-case Payoffs in Quantum Battle of Sexes Game; quant-ph/0110096.
- [26] Du, J. et al, Nash Equilibrium in the Quantum Battle of Sexes Game; quant-ph/0010050.
- [27] Du, J. et al, Remark On Quantum Battle of The Sexes Game; quant-ph/0103004.
- [28] Toyota, N., Quantization of the stag hunt game and the Nash equilibrium; quant-ph/0307029.

- [29] Iqbal A., Toor A. H., Quantum repeated games, *Physics Letters A* **300** (2002) 541.
- [30] Stohler, M., Fischbach, E., Non-Transitive Quantum Games; quant-ph/0307072.
- [31] Huberman, B. A., Hogg, T., Quantum Solution of Coordination Problems; quant-ph/0306112.
- [32] Iqbal A., Toor A.H., Backwards-induction outcome in a quantum game, *Physical Review A* **65** (2002) 052328.
- [33] Flitney, A. P., Abbott, D., Quantum duels and truels; quant-ph/0305058.
- [34] Lee, C. F., Johnson, N. F., Efficiency and formalism of quantum games, *Physical Review A* **67** (2003) 022311.
- [35] Nielsen, M. A., Chuang, I. L., *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge (2000).
- [36] Plenio, M. B., Vitelli, V., The physics of forgetting: Landauer's erasure principle and information theory, *Contemporary Physics* **42** (2001) 25.
- [37] Penrose, R., *Shadows of the Mind*, Cambridge University Press, Cambridge (1994).
- [38] Piotrowski, E. W., Śładkowski, J., The Merchandising Mathematician Model, *Physica A* **318** (2003) 496.
- [39] Śładkowski, J., Giffen paradoxes in quantum market games, *Physica A* **324** (2003) 234.
- [40] Lambertini, L., Quantum Mechanics and Mathematical Economics are Isomorphic; <http://www.spbo.unibo.it/gopher/DSEC/370.pdf>.
- [41] Arfi, B., Ambivalence of Choice and "Holistic" Strategic Interaction: A Quantum Game-Theoretic Proposal; http://atticus.igs.berkeley.edu/research_programs/ppt/papers/barfi.pdf.
- [42] Piotrowski, E. W., Śładkowski, J., Quantum solution to the Newcomb's paradox; quant-ph/0202074.

- [43] Pati, A. K., Braunstein, S. L., Impossibility of deleting an unknown quantum state, *Nature* **404** (2000) 164.
- [44] Horodecki, M., Horodecki, R., De, A. S., Sen, U., No-deleting and no-cloning principles as consequences of conservation of quantum information; quant-ph/0306044.
- [45] Smith, M. J., *Evolution and the Theory of Games*, Cambridge University, New York 1982.
- [46] Iqbal A., Toor A. H., Quantum mechanics gives stability to a Nash equilibrium, *Physical Review A* **65** (2002) 022306.
- [47] Iqbal A., Toor A. H., Darwinism in quantum systems?, *Physics Letters A* **294** (2002) 261.
- [48] Iqbal A., Toor A.H., Entanglement and Dynamic Stability of Nash Equilibria in a Symmetric Quantum Game, *Physics Letters A* **286** (2001) 245.
- [49] Kauffman, L. H., Biologic; quant-ph/0204007.
- [50] Patel, A., Quantum Algorithms and the Genetic Code, *Pramana* **56** (2001) 367.
- [51] Patel, A., Testing Quantum Dynamics in Genetic Information Processing, *Journal of Genetics* **80** (2001) 39.
- [52] Home, D., Chattopadhyaya, R., Determination of When an Outcome is Actualised in a Quantum Measurement using DNA — Photolyase System; . quant-ph/9903036.
- [53] Glimcher, P. W., Decisions, Decisions, Decisions: Choosing a Biological Science of Choice, *Neuron* **36** (2002) 323.
- [54] Glimcher, P. W., *Decisions, Uncertainty, and the Brain: The Science of Neuroeconomics*, MIT Press, Cambridge (2003).
- [55] Piotrowski, E. W., Ślaskowski, J., Quantum computer: an appliance for playing market games; quant-ph/0305017.

- [56] Collum, G., Systems of logical systems: neuroscience and quantum logic, *Foundations of Science* **7** (2002) 49.
- [57] Keyl, M., Fundamentals of Quantum Information Theory, *Physics Reports* **369** (2002) 431.
- [58] Fitzi, M., Gisin, N., Maurer, U., A Quantum solution to the Byzantine agreement problem; quant-ph/0107127.
- [59] Piotrowski, E. W., Fixed Point Theorem for Simple Quantum Strategies in Quantum Market Games, *Physica A* **324** (2003) 196.
- [60] Koenig, R., Maurer, U., Renner, R., On the Power of Quantum Memory; quant-ph/0305154.
- [61] see for example:
http://www.claymath.org/Popular_Lectures/Minesweeper/.
- [62] Deutsch, D., Quantum Theory of Probability and Decisions, *Proceedings of the Royal Society of London A* **455** (1999) 3129.
- [63] Wallace, D., Quantum theory of Probability and Decision Theory, Revisited; quant-ph/0211104.
- [64] Kashefi, E., Kent, A., Vedral, V., Banaszek, K., *Physical Review A* **65** (2002) 050304.
- [65] Kobayashi, H., Matsumoto, K., Yamakami, T., Quantum Merlin-Arthur Proof Systems: Are Multiple Merlins More Helpful to Arthur?; quant-ph/0306051.
- [66] Knill, E., Quantum Randomness and Nondeterminism; quant-ph/9610012.
- [67] Watrous, J., Quantum statistical zero-knowledge; quant-ph/0202111.
- [68] Grib, A., Parfionov, G., Can the game be quantum?; quant-ph/0206178.
- [69] Grib, A., Parfionov, G., Macroscopic quantum game, preprint quant-ph/0211068.
- [70] Piotrowski, E. W., Ślaskowski, J., Quantum Bargaining Games, *Physica A* **308** (2002) 391.

- [71] Pietarinen, A., Quantum Logic and Quantum Theory in a Game-Theoretic Perspective, *Open Systems and Information Dynamics* **9** (2002) 273.
- [72] Goldenberg, L., Vaidman, L., Wiesner, S., Quantum Gambling, *Physical Review Letters* **82** (1999) 3356.
- [73] Zhang, S.-Y. et al, Optical Realization of Quantum Gambling Machine; quant-ph/0001008.
- [74] Hwang, W.-Y., Ahn, D., Hwang, S. W. *Physical Review A* **64** 2001 064302.
- [75] Hwang, W.-Y., Matsumoto, K., Quantum Gambling Using Three Nonorthogonal States, *Physical Review A* **66** (2002) 052311.
- [76] Witte, F. M. C., Quantum 2-player gambling and correlated pay-off; quant-ph/0207085.
- [77] Segre, G., Law of Excluded Quantum Gambling Strategies; quant-ph/0104080.
- [78] Pitowsky, I., Betting on the Outcomes of Measurements: A Bayesian Theory of Quantum Probability, preprint quant-ph/0208121.
- [79] Deutsch, D., *The Fabric of Reality*, Penguin Putnam Incorporated, New York (1998).
- [80] Mendes, R. V., Quantum games and social norms; preprint quant-ph/0208167.
- [81] Velupillai, K., *Computable Economics, the Arne Ryde Memorial Lectures*, Oxford University Press, Oxford (2000).
- [82] Popper, K. R., *The Logic of Scientific Discovery*, Harper Torch Books, New York (1968).
- [83] Hintikka, J., *Logic, language-games and information*, Clarendon Press, Oxford (1973).

- [84] Horodecki, R., Horodecki, M., Horodecki, P., Quantum information isomorphism: beyond the dilemma of Scylla of ontology and Charybdis of instrumentalism, to appear in special issue of the IBM Journal of Research and Development; quant-ph/0305024.