

Quantum Game Theory in Finance

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Abstract

This is a short review of the background and recent development in quantum game theory and its possible application in economics and finance. The intersection of science and society is also discussed. The review is addressed to non-specialists.

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1 Introduction

One hundred years ago, a single concept changed our view of the world forever: quantum theory was born [1]. Contemporary technology is based on implementation of quantum phenomena as a result of this seminal idea.

Regardless of the successes of quantum physics and the resulting quantum technology social sciences persist in classical paradigm what in some aspects can be considered as an obstacle to unification of science in the quantum domain. Quantum theory is up to now the only scientific theory that requires the observer to take into consideration the usually neglected influence of the method of observation on the result of observation. Full and absolutely objective information about the investigated phenomenon is impossible and this is a fundamental principle of Nature and does not result from deficiency in our technology or knowledge. Now, this situation is being changed in a dramatic way. Fascinating results of quantum cryptography, that preceded public key cryptography [2] although not duly appreciated at its infancy, caused that quantum information processing is currently expanding its domain. Various proposals of applying quantum-like models in social sciences and economics has been put forward [3]-[8]. It seems that the numerous acquainted with quantum theory physicists who have recently moved to finance can cause an evolutionary change in the paradigm of methods of mathematical finance. In a quantum world we can explore plenty of parallel simultaneous evolutions of the system and a clever final measurement may bring into existence astonishing and classically inaccessible solutions [8]-[11]. The price we are to pay consists in securing perfect discretion to parallel evolution: any attempt (intended or not) at tracing the system inevitably destroys the desirable quantum effects. Therefore we cannot expect that all quantum aspects can be translated and explained in classical terms [12] (if such a reinterpretation was possible the balance could be easily redressed). Attention to the very physical aspects of information processing revealed new perspectives of computation, cryptography and communication methods. In most of the cases quantum description of the system provides advantages over the classical situation. One should be not surprised that game theory, the study of (rational) decision making in conflict situations, has quantum counterpart. Indeed, games against nature [13] include those for which nature is quantum mechanical. Does quantum theory offer more subtle ways of playing games? Game theory considers strategies that are probabilistic mixtures of pure strategies. Why cannot they be intertwined in a more complicated way, for example interfered or entangled? Are there situations in which quantum theory can enlarge the set of possible strategies? Can quantum strategies be more successful than classical ones? All these questions have positive and sometimes bewildering answers [14, 8]. There are genuine quantum games, that

is games that can be defined and played only in a sophisticated quantum environment. Some of these quantum games could be played only in physical laboratories but technological development can soon change this situation (the most interesting examples emerge from cryptography). Some classical games can be redefined so that quantum strategies can be adopted [15]-[18]. This is ominous because someone can take the advantage of new (quantum) technology if we are not on alert [15, 8]. We should warn the reader that quantum games are games in the classical sense but to play a quantum game may involve sophisticated technology and therefore theoretical analysis of the game requires knowledge of physical theories and phenomena necessary for its implementation. This fact is often overlooked and quantum game theory is wrongly put in sort of opposition to (classical) game theory. Recently, in a series of papers [6, 19, 20] the present authors described market phenomena in terms of quantum game theory. Agents adopting quantum strategies can make profits that are beyond the range of classical markets. Quantum approach shed new light on well known paradoxes [7, 21] and computational complexity of economics [22, 23]. Besides the properties of Nature discovered by human beings there is a whole universe of phenomena and appliances created by mankind. Therefore the question if present day markets reveal any (observable) quantum properties, although interesting, is secondary to our main problem of finding out if genuine quantum markets would ever come into existence. Quantum theory offers a new paradigm that is able to produce a unified description of reality. This paper is organized as follows. First, we present some basic ideas of quantum games. Then we describe quantum market games and review their attractive properties. Finally we present our personal view of the further development and possible applications of this field of research.

2 Quantum market games

As we have said in the Introduction, quantum game theory investigates conflict situations involving quantum phenomena. Therefore it exploits the formalism of quantum theory. In this formalism strategies are vectors (called states) in some Hilbert space and can be interpreted as superpositions of trading decisions. Tactics and moves are performed by unitary transformations on vectors in the Hilbert space (states). The idea behind using quantum games is to explore the possibility of forming linear combination

of amplitudes that are complex Hilbert space vectors (interference, entanglement [8]) whose squared absolute values give probabilities of players actions. It is generally assumed that a physical observable (e.g. energy, position), defined by the prescription for its measurement, is represented by a linear Hermitian operator. Any measurement of an observable produces with some probability an eigenvalue of the operator representing the observable. This probability is given by the squared modulus of the coordinate corresponding to this eigenvalue in the spectral decomposition of the state vector describing the system. This is often an advantage over classical probabilistic description where one always deals directly with probabilities. The formalism has potential applications outside physical laboratories [3]-[6]. But how to describe complex games with say unlimited number of players or non-constant pay-offs. There are several possible ways of accomplishing this task. We have proposed a generalization of market games to the quantum domain in Ref. [6]. In our approach spontaneous or institutionalized market transactions are described in terms of projective operation acting on Hilbert spaces of strategies of the traders. Quantum entanglement is necessary (non-trivial linear combinations of vectors-strategies have to be formed) to strike the balance of trade. This approach predicts the property of undivisibility of attention of traders (no cloning theorem) and unifies the English auction with the Vickrey's one attenuating the motivation properties of the later [24]. Quantum strategies create unique opportunities for making profits during intervals shorter than the characteristic thresholds for an effective market (Brownian motion) [24]. Although the effective market hypothesis assumes immediate price reaction to new information concerning the market the information flow rate is limited by physical laws such as the constancy of the speed of light. Entanglement of states allows to apply quantum protocols of super-dense coding [11] and get ahead of "classical trader". Besides, quantum version of the famous Zeno effect [11] controls the process of reaching the equilibrium state by the market. Quantum arbitrage based on such phenomena seems to be feasible. Interception of profitable quantum strategies is forbidden by the impossibility of cloning of quantum states. There are apparent analogies with quantum thermodynamics that allow to interpret market equilibrium as a state with vanishing financial risk flow. Euphoria, panic or herd instinct often cause violent changes of market prices. Such phenomena can be described by non-commutative quantum mechanics. A simple tactics that maximize the trader's profit on an effective market follows from the model: *accept profits equal or greater than the*

one you have formerly achieved on average [25].

We were led to these conclusions by consideration of the following facts:

- error theory: second moments of a random variable describe errors,
- H. Markowitz's portfolio theory,
- L. Bachelier's theory of options: the random variable $q^2 + p^2$ measures joint risk for a stock buying–selling transaction (and Merton & Scholes works that gave them Nobel Prize in 1997).

We have defined canonically conjugate Hermitian operators (observables) of demand \mathcal{Q}_k and supply \mathcal{P}_k corresponding to the variables q and p characterizing strategy of the k -th player. These operators act on the player's strategy states $|\psi\rangle^1$ that have two important representations $\langle q|\psi\rangle$ (demand representation) and $\langle p|\psi\rangle$ (supply representation) where q and p are logarithms of prices at which the player is buying or selling, respectively [11, 26]. This led us to the following definition of the observable that we call *the risk inclination operator* [26]²:

$$H(\mathcal{P}_k, \mathcal{Q}_k) := \frac{(\mathcal{P}_k - p_{k0})^2}{2m} + \frac{m\omega^2(\mathcal{Q}_k - q_{k0})^2}{2},$$

where $p_{k0} := \frac{k\langle\psi|\mathcal{P}_k|\psi\rangle_k}{k\langle\psi|\psi\rangle_k}$, $q_{k0} := \frac{k\langle\psi|\mathcal{Q}_k|\psi\rangle_k}{k\langle\psi|\psi\rangle_k}$, $\omega := \frac{2\pi}{\theta}$. θ denotes the characteristic time of transaction [25, 26] which is, roughly speaking, an average time spread between two opposite moves of a player (e. g. buying and selling the same commodity). The parameter $m > 0$ measures the risk asymmetry between buying and selling positions. Analogies with quantum harmonic oscillator allow for the following characterization of quantum market games. One can introduce an analogue of the Planck constant, h_E , that describes

¹We use the standard Dirac notation. The symbol $|\rangle$ with a letter ψ in it denoting a vector parameterized by ψ is called a *ket*; the symbol $\langle |$ with a letter in it is called a *bra*. Actually a *bra* is a dual vector to the corresponding *ket*. Therefore scalar products of vectors take the form $\langle\phi|\psi\rangle$ (*bracket*) and the expectation value of an operator A in the state $|\psi\rangle$ is given by $\langle\psi|A\psi\rangle$. A common abuse of this convention consist in denoting the wave function $\psi(p)$ as $\langle p|\psi\rangle$. (A wave functions is a vector in Hilbert space of square integrable functions and one associates with the variable p an eigenvector $|p\rangle$.)

²The reader that is familiar with the rudiments of quantum mechanics would certainly notice that this operator is nothing else then the hamiltonian for quantum harmonic oscillator.

the minimal inclination of the player to risk, $[\mathcal{P}_k, \mathcal{Q}_k] = \frac{i}{2\pi}h_E$. As the lowest eigenvalue of the positive definite operator H is $\frac{1}{2}\frac{h_E}{2\pi}\omega$, h_E is equal to the product of the lowest eigenvalue of $H(\mathcal{P}_k, \mathcal{Q}_k)$ and 2θ . 2θ is in fact the minimal interval during which it makes sense to measure the profit. In a general case the operators \mathcal{Q}_k do not commute because traders observe moves of other players and often act accordingly. One big bid can influence the market at least in a limited time spread. Therefore it is natural to apply the formalism of noncommutative quantum mechanics where one considers

$$[x^i, x^k] = i\Theta^{ik} := i\Theta \epsilon^{ik}. \quad (3)$$

The analysis of harmonic oscillator in more than one dimension [27] imply that the parameter Θ modifies the constant $\hbar_E \rightarrow \sqrt{\hbar_E^2 + \Theta^2}$ and the eigenvalues of $H(\mathcal{P}_k, \mathcal{Q}_k)$ accordingly. This has the natural interpretation that moves performed by other players can diminish or increase one's inclination to take risk. Encouraged by that we asked the question *Provided that an all-purpose quantum computer is built, how would a market cleared by a quantum computer perform?* To find out we have to consider quantum games with unlimited and changing number of players. A possible approach is as follows. If a game allows a great number of players in it is useful to consider it as a two-players game: the k -th trader against the Rest of the World (RW). Any concrete algorithm \mathcal{A} should allow for an effective strategy of RW (for a sufficiently large number of players the single player strategy should not influence the form of the RW strategy). Let the real variable q

$$q := \ln c - E(\ln c)$$

denotes the logarithm of the price at which the k -th player can buy the asset \mathfrak{G} shifted so that its expectation value in the state $|\psi\rangle_k$ vanishes. The expectation value of x is denoted by $E(x)$. The variable p

$$p := E(\ln c) - \ln c$$

describes the situation of a player who is supplying the asset \mathfrak{G} according to his strategy $|\psi\rangle_k$. Supplying \mathfrak{G} can be regarded as demanding $\$\$ at the price c^{-1} in the $1\mathfrak{G}$ units and both definitions are equivalent. Note that we have defined q and p so that they do not depend on possible choices of the units for \mathfrak{G} and $\$$. For simplicity we will use such units that $E(\ln c) = 0$. The strategies $|\psi\rangle_k$ belong to Hilbert spaces H_k . The state of the game $|\Psi\rangle_{in} :=$

$\sum_k |\psi\rangle_k$ is a vector in the direct sum of Hilbert spaces of all players, $\oplus_k H_k$. We will define canonically conjugate hermitian operators of demand \mathcal{Q}_k and supply \mathcal{P}_k for each Hilbert space H_k analogously to their physical position and momentum counterparts. This can be justified in the following way. Let $\exp(-p)$ be a definite price, where p is a proper value of the operator \mathcal{P}_k . Therefore, if one have already declared the will of selling exactly at the price $\exp(-p)$ (the strategy given by the proper state $|p\rangle_k$) then it is pointless to put forward any opposite offer for the same transaction. The capital flows resulting from an ensemble of simultaneous transactions correspond to the physical process of measurement. A transaction consists in a transition from the state of traders strategies $|\Psi\rangle_{in}$ to the one describing the capital flow state $|\Psi\rangle_{out} := \mathcal{T}_\sigma|\Psi\rangle_{in}$, where $\mathcal{T}_\sigma := \sum_{k_d} |q\rangle_{k_d k_d} \langle q| + \sum_{k_s} |p\rangle_{k_s k_s} \langle p|$ is the projective operator defined by the division σ of the set of traders $\{k\}$ into two separate subsets $\{k\} = \{k_d\} \cup \{k_s\}$, the ones buying at the price $e^{q_{k_d}}$ and the ones selling at the price $e^{-p_{k_s}}$ in the round of the transaction in question. The role of the algorithm \mathcal{A} is to determine the division σ of the market, the set of price parameters $\{q_{k_d}, p_{k_s}\}$ and the values of capital flows. The later are settled by the distribution

$$\int_{-\infty}^{\ln c} \frac{|\langle q|\psi\rangle_k|^2}{k\langle\psi|\psi\rangle_k} dq$$

which is interpreted as the probability that the trader $|\psi\rangle_k$ is willing to buy the asset \mathfrak{G} at the transaction price c or lower [25]. In an analogous way the distribution

$$\int_{-\infty}^{\ln \frac{1}{c}} \frac{|\langle p|\psi\rangle_k|^2}{k\langle\psi|\psi\rangle_k} dp$$

gives the probability of selling \mathfrak{G} by the trader $|\psi\rangle_k$ at the price c or greater. These probabilities are in fact conditional because they describe the situation after the division σ is completed. If one considers the RW strategy it make sense to declare its simultaneous demand and supply states because for one player RW is a buyer and for another it is a seller. To describe such situation it is convenient to use the Wigner formalism ³ [28]. The pseudo-probability $W(p, q)dpdq$ on the phase space $\{(p, q)\}$ known as the Wigner

³Actually, this approach consists in allowing pseudo-probabilities into consideration. From the physical point of view this is questionable but for our aims its useful, c.f. the discussion of the Giffen paradox [21].

function is given by

$$\begin{aligned} W(p, q) &:= h_E^{-1} \int_{-\infty}^{\infty} e^{i\hbar_E^{-1} px} \frac{\langle q + \frac{x}{2} | \psi \rangle \langle \psi | q - \frac{x}{2} \rangle}{\langle \psi | \psi \rangle} dx \\ &= h_E^{-2} \int_{-\infty}^{\infty} e^{i\hbar_E^{-1} qx} \frac{\langle p + \frac{x}{2} | \psi \rangle \langle \psi | p - \frac{x}{2} \rangle}{\langle \psi | \psi \rangle} dx, \end{aligned}$$

where the positive constant $h_E = 2\pi\hbar_E$ is the dimensionless economical counterpart of the Planck constant. Recall that this measure is not positive definite except for the cases presented below. In the general case the pseudo-probability density of RW is a countable linear combination of Wigner functions, $\rho(p, q) = \sum_n w_n W_n(p, q)$, $w_n \geq 0$, $\sum_n w_n = 1$. The diagrams of the integrals of the RW pseudo-probabilities (see Ref. [25])

$$F_d(\ln c) := \int_{-\infty}^{\ln c} \rho(p = \text{const.}, q) dq$$

(RW bids selling at $\exp(-p)$)

and

$$F_s(\ln c) := \int_{-\infty}^{\ln \frac{1}{c}} \rho(p, q = \text{const.}) dp$$

(RW bids buying at $\exp(q)$) against the argument $\ln c$ may be interpreted as the dominant supply and demand curves in the Cournot convention, respectively [25]. Note, that due to the lack of positive definiteness of ρ , F_d and F_s may not be monotonic functions. Textbooks on economics give examples of such departures from the law of supply (work supply) and law of demand (Giffen assets) [29]. We proposed to call an arbitrage algorithm resulting in non positive definite probability densities *a giffen*. The following subsection describe shortly various aspects of quantum markets.

2.1 Quantum Zeno effect

It has been experimentally verified that sufficiently frequent measurement can slow down (accelerate) the dynamics of a quantum process, what is called the *quantum (anti-)Zeno effect* [30]. Analogous phenomenon can be observed in quantum games. If the market continuously measures the same strategy of the player, say the demand $\langle q | \psi \rangle$, and the process is repeated sufficiently often for the whole market, then the prices given by the

algorithm \mathcal{A} do not result from the supplying strategy $\langle p|\psi \rangle$ of the player. The necessary condition for determining the profit of the game is the transition of the player to the state $\langle p|\psi \rangle$ [25]. If, simultaneously, many of the players change their strategies then the quotation process may collapse due to the lack of opposite moves. In this way the quantum Zeno effects explain stock exchange crashes. Effects of this crashes should be predictable because the amplitudes of the strategies $\langle p|\psi \rangle$ are Fourier transforms of $\langle q|\psi \rangle$. Another example of the quantum market Zeno effect is the stabilization of prices of an asset provided by a monopolist.

2.2 Eigenstates of \mathcal{Q} and \mathcal{P}

Let us suppose that the amplitudes for the strategies $\langle q|\psi \rangle_k$ or $\langle p|\psi \rangle_k$ have divergent integrals of their modulus squared. Such states live outside the Hilbert space but have the natural interpretation as the desperate will of the k -th player of buying (selling) of the amount d_k (s_k) of the asset \mathfrak{G} . So the strategy $\langle q|\psi \rangle_k = \langle q|a \rangle = \delta(q, a)$ means, in the case of classifying the player into the set $\{k_d\}$, refusal of buying cheaper than at $c = e^a$ and the will of buying at any price equal or higher than e^a . In the case of a "measurement" in the set $\{k_d\}$ the player declares the will of selling at any price. The above interpretation is consistent with the Heisenberg uncertainty relation. The strategies $\langle q|\psi \rangle_2 = \langle q|a \rangle$ (or $\langle p|\psi \rangle_2 = \langle p|a \rangle$) do not correspond to the RW behaviour because the conditions $d_2, s_2 > 0$, if always satisfied, allow for unlimited profits (the readiness to buy or sell \mathfrak{G} at any price). The appropriate demand and supply functions give probabilities of coming off transactions in a game when the player use the strategy $\langle p|const \rangle$ or $\langle q|const \rangle$ and RW, proposing the price, use the strategy ρ [6, 25]. The authors have analyzed the efficiency of the strategy $\langle q|\psi \rangle_1 = \langle q| - a \rangle$ in a two-player game when RW use the strategy with squared modulus of the amplitude equal to normal distribution [25]. The maximal intensity of the profit [25] is equal to 0.27603 times the variance of the RW distribution function. Of course, the strategy $\langle p|\psi \rangle_1 = \langle p|0, 27603 \rangle$ has the same properties. In such games $a=0.27603$ is a global fixed point of the profit intensity function. This may explain the universality of markets on which a single client facing the bid makes up his/hers mind. Does it mean that such common phenomena have quantal nature? The Gaussian strategy of RW [31] can be parameterized by a temperature-like parameter $T = \beta^{-1}$. Any decrease in profits is only possible by reducing the variance of RW (i.e.

cooling). Market competition is the mechanism responsible for the risk flow that allows the market to attain the "thermodynamical" balance. A warmer market influences destructively the cooler traders and they diminish the uncertainty of market prices.

2.3 Correlated coherent strategies

We will define correlated coherent strategies as the eigenvectors of the annihilation operator \mathcal{C}_k [32]

$$\mathcal{C}_k(r, \eta) := \frac{1}{2\eta} \left(1 + \frac{ir}{\sqrt{1-r^2}} \right) \mathcal{Q}_k + i\eta \mathcal{P}_k,$$

where r is the correlation coefficient $r \in [-1, 1]$, $\eta > 0$. In these strategies buying and selling transactions are correlated and the product of dispersions fulfills the Heisenberg-like uncertainty relation $\Delta_p \Delta_q \sqrt{1-r^2} \geq \frac{\hbar_E}{2}$ and is minimal. The annihilation operators \mathcal{C}_k and their eigenvectors may be parameterized by $\Delta_p = \frac{\hbar_E}{2\eta}$, $\Delta_q = \frac{\eta}{\sqrt{1-r^2}}$, and r . This leads to following form of the correlated Wigner coherent strategy

$$W(p, q) dp dq = \frac{1}{2\pi \Delta_p \Delta_q \sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)} \left(\frac{(p-p_0)^2}{\Delta_p^2} + \frac{2r(p-p_0)(q-q_0)}{\Delta_p \Delta_q} + \frac{(q-q_0)^2}{\Delta_q^2} \right)} dp dq.$$

They are not giffens. It can be shown, following Hudson [33], that they form the set of all pure strategies with positive definite Wigner functions. Therefore pure strategies that are not giffens are represented in phase space $\{(p, q)\}$ by gaussian distributions.

2.4 Mixed states and thermal strategies

According to classics of game theory [34] the biggest choice of strategies is provided by the mixed states $\rho(p, q)$. Among them the most interesting are the thermal ones. They are characterized by constant inclination to risk, $E(H(\mathcal{P}, \mathcal{Q})) = \text{const}$ and maximal entropy. The Wigner measure for the n -th excited state of harmonic oscillator has the form [35]

$$W_n(p, q) dp dq = \frac{(-1)^n}{\pi \hbar_E} e^{-\frac{2H(p, q)}{\hbar_E \omega}} L_n \left(\frac{4H(p, q)}{\hbar_E \omega} \right) dp dq,$$

where L_n is the n -th Laguerre polynomial. The mixed state ρ_β determined by the Wigner measures $W_n dpdq$ weighted by the Gibbs distribution $w_n(\beta) := \frac{e^{-\beta n \hbar_E \omega}}{\sum_{k=0}^{\infty} e^{-\beta k \hbar_E \omega}}$ has the form

$$\begin{aligned}\rho_\beta(p, q) dpdq &:= \sum_{n=0}^{\infty} w_n(\beta) W_n(p, q) dpdq \\ &= \frac{\omega}{2\pi} x e^{-xH(p, q)} \Big|_{x = \frac{2}{\hbar_E \omega} \tanh(\beta \frac{\hbar_E \omega}{2})} dpdq.\end{aligned}$$

So it is a two dimensional normal distribution. It is easy to observe that by recalling that $\frac{1}{1-t} e^{\frac{xt}{t-1}} = \sum_{n=0}^{\infty} L_n(x) t^n$ is the generating function for the Laguerre polynomials. It seems to us that the above distributions should determine the shape of the supply and demand curves for equilibrium markets. There are no giffens on such markets. It would be interesting to investigate the temperatures of equilibrium markets. In contrast to the traders temperatures [31] which are Legendre coefficients and measure "trader's qualities" market temperatures are related to risk and are positive. The Feynman path integrals may be applied to the Hamiltonian to obtain equilibrium quantum Bachelier model of diffusion of the logarithm of prices of shares that can be completed by the Black-Scholes formula for pricing European options [36].

2.5 Quantum auctions and bargaining

After tasting the exotic flavour of quantum market games one may wish to distinguish the class of quantum transactions (q-transactions) that is q-games without institutionalized clearinghouses. This class includes quantum bargaining (q-bargaining) and quantum auctions (q-auction). The participants of a q-bargaining game will be called Alice (A) and Bob (B). We will suppose that they settle on beforehand who is the buyer (Alice) and who is the seller (Bob). A two-way q-bargaining that is a q-bargaining when the last condition is not fulfilled can be treated analogously. Alice enter into negotiations with Bob to settle the price for the transaction. Therefore the proper measuring apparatus consists of the pair of traders in question. In q-auction the measuring apparatus consists of a one side only, the initiator of the auction. We showed [6] that the players strategies can be described in terms of polarizations, that is the states in a two-dimensional Hilbert

space. If the player formulates the conditions of the transaction we say she has the polarization $|1\rangle$ (and is in the state $|\vec{r}\rangle_A = |1\rangle$). In q-bargaining this means that she puts forward the price. In the opposite case, when she decides if the transaction is made or not, we say she has the polarization $|0\rangle$. (She accepts or not the conditions of the proposed transaction.) There is an analogy of the isospin symmetry in nuclear physics which says that nucleon has two polarization states: proton and neutron. The vectors $(|0\rangle, |1\rangle)$ form an orthonormal basis in \mathcal{H}_s , the linear hull of all possible Alice polarization states. The player 1 proposes a price and the player -1 accepts or reject the proposal. Therefore their polarizations are $|0\rangle$ and $|1\rangle$, respectively so the q-bargaining has the polarization $|0\rangle_{-1}|1\rangle_1$ [6, 19]. The transaction in question is accomplished if the obvious rationality condition is fulfilled

$$[\mathfrak{q} + \mathfrak{p} \leq 0],$$

where the convenient Iverson notation [37] is used ($[expression]$ denotes the logical value (1 or 0) of the sentence *expression*) and the parameters $\mathfrak{p} = \ln \mathfrak{c}_{-1}$ and $-\mathfrak{q} = \ln \mathfrak{c}_1$ are random variables corresponding to prices at which the respective players withdraw, the *withdrawal prices*. The variables \mathfrak{p} and \mathfrak{q} describe (additive) profits resulting from price variations. Their probability densities are equal to squared absolute values of the appropriate wave functions $\langle p|\psi \rangle_{-1}$ and $\langle q|\psi \rangle_1$ (that is their strategies). Note that the discussed q-bargaining may result from a situation where several players have intention of buying but they were outbid by the player 1 (his withdrawal price \mathfrak{c}_1 was greater than the other players ones, $\mathfrak{c}_1 > \mathfrak{c}_k$, $k = 2, \dots, N$). This means that all part in the auction behave like fermions (e.g. electrons) and they are subjected to a sort of Pauli exclusion principle according to which two players cannot occupy the same state. This surprising statement consist in noticing that the transaction in question is made only if the traders have opposite polarizations (and even that is not a guarantee of the accomplishment). The fermionic character of q-bargaining parts was first noted in [6] in a slightly different context. If at the outset of the auction there are several bidding players then the rationality condition takes the form

$$[\mathfrak{q}_{\min} + \mathfrak{p} \leq 0]$$

where $\mathfrak{q}_{\min} := \min_{k=1, \dots, N} \{\mathfrak{q}_k\}$ is the logarithm of the highest bid multiplied by -1 . According to Ref. [6] the probability density of making the transaction

with the k -th buyer at the price $c_k = e^{-q_k}$ is given by

$$dq_k \frac{|\langle q_k | \psi_k \rangle|^2}{\langle \psi_k | \psi_k \rangle} \prod_{\substack{m=1 \\ m \neq k}}^N \int_{-\infty}^{\infty} dq_m \frac{|\langle q_m | \psi_m \rangle|^2}{\langle \psi_m | \psi_m \rangle} \int_{-\infty}^{\infty} dp \frac{|\langle p | \psi_{-1} \rangle|^2}{\langle \psi_{-1} | \psi_{-1} \rangle} \quad [q_k = \min_{n=1, \dots, N} \{q_n\}] \quad [q_k + p \leq 0]. \quad (1)$$

The seller is not interested in making the deal with any particular buyer and the unconditional probability of accomplishing the transaction at the price c is given by the sum over $k = 1, \dots, N$ of the above formula with $q_k = -\ln c$. If we neglect the problem of determining the probability amplitudes in (1) we easily note that the discussed q-bargaining is in fact an English auction (first price auction), so popular on markets of rare goods. It is interesting to note that the formula (1) contains wave functions of payers who were outbid before the end of the bargaining (cf the Pauli exclusion principle). The probability density of "measuring" of a concrete value q of the random variable q characterizing the player, according to the probabilistic interpretation of quantum theory, is equal to the squared absolute value of the normalized wave function describing his strategy

$$\frac{|\langle q | \psi_k \rangle|^2}{\langle \psi_k | \psi_k \rangle} dq.$$

Physicists normalize wave functions because conservation laws require that. Therefore the trivial statement that if a market player may be persuaded into striking a deal or not is a matter of price alone, corresponds to the physical fact that a particle cannot vanish without any trace. The analysis of an English q-auction with reversed roles that is with selling bidders is analogous. The case when the polarization of the q-auction is changed to $|1\rangle_{-1}|0\rangle_1$ is more interesting. In this case the player -1 reveals her withdrawal price and the player 2 accepts it (as the rest of the players do) or not. Such an auction is known as the Vickrey's auction (or the second price auction). The winner is obliged to pay the second in decreasing order price from all the bids (and the withdrawal price of the player -1). In the quantum approach English and Vickrey's auctions are only special cases of a phenomenon called q-auction. In the general case both squared amplitudes $|\langle 0_{-1} 1_1 | 0_{-1} 1_1 \rangle|^2$ and $|\langle 1_{-1} 0_1 | 1_{-1} 0_1 \rangle|^2$ are non-vanishing so we have to consider them with weights corresponding to these probabilities. Such a general q-auction does not have counterparts on the real markets. It should be very interesting to analyse the motivation properties of q-auctions eg finding out

when the best strategy is the one corresponding to the player's valuation of the good. If we consider only positive definite probability measures then the bidder gets the highest profits in Vickrey's auction using strategies with public admission of his valuation of the auctioned good. But it might not be so for giffen strategies because positiveness of measures is supposed in proving the incentive character of Vickrey's auctions [38]. The presence of giffens on real markets might not be so abstract as it seems to be. Captain Robert Giffen who supposedly found additive measures not being positive definite but present on markets in the forties of the XIX century [39] probably got ahead of physicists in observing quantum phenomena. Such departures from the demand law, if correctly interpreted, does not cause any problem neither for adepts nor for beginners. Employers have probably always thought that work supply as function of payment is scarcely monotonous. The distinguished by their polarization first and second price auctions have analogues in the Knaster solution to the pragmatic fair division problem (with compensatory payments for indivisible parts of the property) [40]. Such a duality might also be found in election systems that as auctions often take the form of procedures of solving fair division problems [41]. It might happen that social frustrations caused by election systems would encourage us to discuss such topics.

3 Conclusions

The commonly accepted universality of quantum theory should encourage physicist in looking for traces of quantum world in social phenomena. We envisage markets cleared by quantum computer. We hope that the sketchy analysis presented above would allow the reader to taste the exotic flavours of quantum markets. A quantum theory of markets provides new tools that can be used to explain of the very involved phenomena including interference of (quantum) strategies [42] and diffusion of prices [43]. The research into the quantum nature of games may offer solutions to very intriguing paradoxes present in philosophy and economics. For example, the Newcomb's paradox analyzed in Ref. [7] suggests various ones. There are quantum games that live across the border of our present knowledge. For example, consider some classical or quantum problem X . Let us define the game $kXcl$: you win if and only if you solve the problem (perform the task) X given access to only k bits of information. The quantum counterpart

reads: solve the problem X on a quantum computer or other quantum device given access to only k bits of information. Let us call the game $kXcl$ or kXq interesting if the corresponding limited information-tasks are feasible. Let $OckhamXcl$ ($OckhamXq$) denotes the minimal k interesting game in the class $kXcl$ (kXq). Authors of the paper [44] described the game played by a market trader who gains the profit P for each bit (qubit) of information about her strategy. If we denote this game by MP then $OckhamM\frac{1}{2}cl = 2M\frac{1}{2}cl$ and for $P > \frac{1}{2}$ the game $OckhamMPcl$ does not exist. They also considered the more effective game $1M\frac{2+\sqrt{2}}{4}q$ for which $OckhamM\frac{2+\sqrt{2}}{4}q \neq 1M\frac{2+\sqrt{2}}{4}q$ if the trader can operate on more than one market. This happens because there are entangled strategies that are more profitable [45]. There are a lot of intriguing questions that can be asked, for example for which X the meta-game $Ockham(OckhamXq)cl$ can be solved or when, if at all, the meta-problem $Ockham(OckhamXq)q$ is well defined problem. Such problems arise in quantum memory analysis [46]. We would like to stress that this field of research undergoes an eventful development. Therefore now it is difficult to predict which results would turn out to be fruitful and which would have only marginal effect.

Recent research on the (quantum) physical aspects of information processing should result in a sort of *total quantum paradigm* and we dare to say that quantum game approach became sooner or later a dominant one. Therefore we envisage markets cleared by quantum algorithms (computers)⁴.

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⁴Let us quote the Editor's Note to Complexity Digest 2001.27(4) (<http://www.comdig.org>): "It might be that while observing the due ceremonial of everyday market transaction we are in fact observing capital flows resulting from quantum games eluding classical description. If human decisions can be traced to microscopic quantum events one would expect that nature would have taken advantage of quantum computation in evolving complex brains. In that sense one could indeed say that quantum computers are playing their market games according to quantum rules".

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