Quantum English Auctions

(RePEc:sla:eakjkl:6 3-VIII-2001)

Edward W. Piotrowski
Institute of Theoretical Physics, University of Białystok,
Lipowa 41, Pl 15424 Białystok, Poland
e-mail: ep@alpha.uwb.edu.pl
Jan Sładkowski
Institute of Physics, University of Silesia,
Uniwersytecka 4, Pl 40007 Katowice, Poland
e-mail: sladk@us.edu.pl

Abstract

We continue the analysis of quantum-like description of markets and economics. The approach has roots in the recently developed quantum game theory and quantum computing. The present paper is devoted to quantum English auction which are a special class of quantum market games. The approach allows to calculate profit intensities for various possible strategies.

PACS numbers: 02.50.Le, 03.67.-a, 03.65.Bz

1 Introduction

Recent research on quantum computation and quantum information allowed to extend the scope game theory for the quantum world [1]-[4]. We showed how quantum game theory may be used for describing financial market phenomena [5, 6]. The purpose of this paper is to extent the previous results to incorporate also quantum version of English auctions. Such a

generalization is desirable because auctions prevail among market games and we think that quantum-like approach provide us with more precise models of market phenomena than the standard ones based on probability theory. The quantum-like description of market phenomena has a remarkable chance of gaining favourable reception from the experts. On the other hand only thorough investigation may reveal if economics already is in or would ever enter the domain of quantum theory. Quantum computation is on the verge of being recognized as an autonomous scientific discipline and efforts to unify social and physical phenomena should not cause astonishment [7]. It might be that while observing the due ceremonial of everyday market transaction we are in fact observing capital flows resulting from quantum games eluding classical description. " If human decisions can be traced to microscopic quantum events one would expect that nature would have taken advantage of quantum computation in evolving complex brains. In that sense one could indeed say that quantum computers are playing their market games according to quantum rules" [8] .
In the following sections we consider quantum English auctions and analyze possible profits gained under various conditions. Vickrey's auctions and various generalizations would be presented in following papers.

2 Quantum bargaining with one-side bidding

Let us consider a particular case of quantum bargaining (q-bargaining) [5, 6] in which the first player, denoted by -1 for future convenience, sells a definite amount of some good and the second one, denoted by 1 want to buy the good in question. The player 1 proposes a price and the player -1 accept or reject the proposal. Their polarizations [6] are $|0\rangle$ and $|1\rangle$, respectively so the q-bargaining has the polarization $|0\rangle_{-1}|1\rangle_{1}$. The transaction in question is accomplished if the obvious rationality condition is fulfilled

$$[\mathfrak{q} + \mathfrak{p} \leq 0], \tag{1}$$

where the convenient Iverson notation [9] is used ($[expression]$ denotes the logical value (1 or 0) of the sentence $expression$) and the parameters $\mathfrak{p} = \ln \mathfrak{c}_{-1}$ and $-\mathfrak{q} = \ln \mathfrak{c}_{1}$ are random variables corresponding to prices at which the respective players withdraw, the *withdrawal prices*. The random variables $\mathfrak{p}$ and $\mathfrak{q}$ describe additively profits resulting from price variations. Their probability densities are equal to squared absolute values of the appropriate wave functions $\langle p|\psi\rangle_{\text{-}1}$ and $\langle q|\psi\rangle_1$ (that is their strategies). Note that the discussed q-bargaining may result from a situation where several players have intention of buying but they were outbid by the player 1 (his withdrawal price $\mathfrak{c}_1$ was greater than the other players ones, $\mathfrak{c}_1 > \mathfrak{c}_k,\ k = 2,\ldots,N$). This means that all part in the auction are fermions and they are subjected to the Pauli exclusion principle according to which two players cannot occupy the same state. The fermionic character of q-bargaining parts first noted in [5] in a slightly different context. If at the outset of the auction there are several bidding players then the rationality condition takes the form

$$[\mathfrak{q}_{\min} + \mathfrak{p} \leq 0] \tag{2}$$

where $\mathfrak{q}_{\min} := \min\limits_{k=1,\ldots,N}\{\mathfrak{q}_k\}$ is the logarithm of the highest bid multiplied by -1. The probability density of making the transaction with the k-th buyers at the price $c_k = \mathrm{e}^{-q_k}$ is according to Ref. [5] given by

$$dq_k\ \frac{|\langle q_k|\psi_k\rangle|^2}{\langle\psi_k|\psi_k\rangle}\prod_{\substack{m=1\\ m\neq k}}^{N}\int_{-\infty}^{\infty}dq_m\frac{|\langle q_m|\psi_m\rangle|^2}{\langle\psi_m|\psi_m\rangle}\int_{-\infty}^{\infty}dp\ \frac{|\langle p|\psi_{\text{-}1}\rangle|^2}{\langle\psi_{\text{-}1}|\psi_{\text{-}1}\rangle}\ [\ q_k = \min_{n=1,\ldots,N}\{q_n\}\ ]\ [q_k{+}p \leq 0]. \tag{3}$$

The seller is not interested in making the deal with any particular buyer and the unconditional probability of accomplishing the transaction at the price c is given by the sum over $k=1,\ldots,N$ of the above formula with $q_k = -\ln c$. If we neglect the problem of determining the probability amplitudes in (3) we easily note that the discussed q-bargaining is in fact the English auction (first price auction) so popular on markets of rare goods. From the quantum context it is interesting to note that the formula (3) contains wave functions of payers who were outbid before the end of the bargaining (cf the Pauli exclusion principle). The probability density of "measuring" of a concrete value q of the random variable $\mathfrak{q}$ characterizing the player, according to the probabilistic interpretation of quantum theory, is equal to the squared absolute value of the normalized wave function describing his strategy

$$\frac{|\langle q|\psi_k\rangle|^2}{\langle\psi_k|\psi_k\rangle}\,dq\ . \tag{4}$$

Physicists normalize wave functions because conservation laws require that. Therefore the trivial statement that if a market player may be persuaded into making a deal or not is a matter of price alone, corresponds to the physical fact that a particle cannot vanish without any trace.

3 Quantum English auction with a dominating bidder

The most frequent scenario of an English auction is the one with public reserve price (bids lower than the reserve price are rejected). The quantum version of such an action may defined as the auction when measures are cut from one side and the player -1 does not fix his withdrawal price ($|\langle q|\psi_{\text{-}1}\rangle|^2 = \delta(q-q')$). We cannot identify withdrawal and the reserve prices in quantum approach because this would result in contradiction because it would entangle the reserve price with the players -1' polarization which forbiden by the Pauli exclusion principle (both players would wind up in the same polarization state before settlement of the bargain).
We restrict our analysis to quantum English auctions during which the player -1 has fixed withdrawal price $\mathfrak{c} = \mathrm{e}^{p'}$. The corresponding probability measure is equal to $|\langle p|\psi_{\text{-}1}\rangle|^2 dp = \delta(p - p')dp$. We will also suppose that players are allowed to use mixed strategies. In that case the squared absolute values of probability amplitudes $|\langle q_k|\psi\rangle|^2$ in (3) should be replaced by appropriate convex linear combination $\eta(q_k)$.
If for some $k = k'$ the formula (3) might by replaced by the measure

$$[q_{k'} + p' \leq 0]\ \eta(q_{k'})\ dq_{k'} \tag{5}$$

then the auction in question reduces the merchandising mathematician model [5, 6] that is to q-bargaining with the polarization $|0\rangle_{\text{-}1}|1\rangle_1$ and the player -1 strategy being a proper state of the operator of supply $\mathcal{P}$ or operator of demand $\mathcal{Q}$ if she is selling or buying, respectively. This may happen if the measure of the set of events for which $\mathfrak{q}_{k'} \neq \min_{n=1,\ldots,N}\{\mathfrak{q}_n\}$ is negligible eg k'-th player offers are to high for the rest of participants.

4 Quantum English auction with identical strategies of bidders

Let us now consider the class of English auctions with all N buyers having the same density of distribution of the logarithm of the withdrawal price, $\eta(q)$, which may be interpreted as the strategy of a equilibrium market with the mean value of the withdrawal price equal to zero (one may always find appropriate currency units). The formula (3) reduces to

$$[q+p'\leq 0]\;\eta(q)\;\Big(\int_q^{\infty}\eta(r)\;dr\;\Big)^{N-1}dq \tag{6}$$

because the probability of success in the auction with price belonging to $[\mathrm{e}^{-q},\mathrm{e}^{-q}(1+dq)]$ does not depend on the player. The random variable $-\mathfrak{q}$ represents the profits measured by the compound rate of return achieved by the player -1 in the auction with respect to the average market price of the good being sold. To measure the profits of the seller is sufficient to notice that her situation is identical to q-bargaining with fixed polarization. Her abstract opponent being the Rest of the World [5] might accomplish the bargaining by bidding price whose logarithm with reversed sing is a random variable $\mathfrak{q}'$ with the distribution equal to N times the distribution (6) that is the function $\mathfrak{q}' := \min\{\mathfrak{q}_1,\ldots,\mathfrak{q}_N\}$ (1th-order statistics [10]). The profit intensity of the seller takes the form [11]

$$\rho_N(p') := \frac{E(-[\mathfrak{q}'+p'\leq 0]\;\mathfrak{q}')}{E(\mathfrak{t})} = \frac{-\int\limits_{-\infty}^{-p'} q\;\eta(q)\;\big(\int\limits_{q}^{\infty}\eta(r)\;dr\big)^{N-1}dq}{N^{-1}+\int\limits_{-\infty}^{-p'}\eta(q)\;\big(\int\limits_{q}^{\infty}\eta(r)\;dr\big)^{N-1}dq} \tag{7}$$

where $\mathfrak{t}$ is the random variable describing time needed by the player -1 an average profit $E(-[\mathfrak{q}'+p'\leq 0]\;\mathfrak{q}')$ (with a fixed withdrawal price p'). Let us recall that (7) has a remarkable property of attaining it maximal value at a fixed point, that, if $\mathfrak{q}$ has normal distribution, is a contraction almost everywhere. Normal distributions play a special role in quantum market games models because they exhaust the class of positive definite pure strategies [5, 6]. They describe also equilibrium markets. Therefore til the end of next paragrph we will suppose that $\eta(q)$ is a normal distribution. If $\rho_N(p')$ is a contraction then the opponent of bidders may use a natural method of maximization of her profit intensity. The method consists in repeated corrections of the withdrawal price up to the value equal to mean profit intensity [5, 6, 11]. The knowledge of the character of the distribution $\eta(q)$ is not necessary. But if the number of bidders is big the same result may be achieved by setting the withdrawal price to zero ($p'=-\infty$). Fig. 1 presents the fall in values of the function $\rho_N(p')$ from maximum to $\rho_N(-\infty)$.

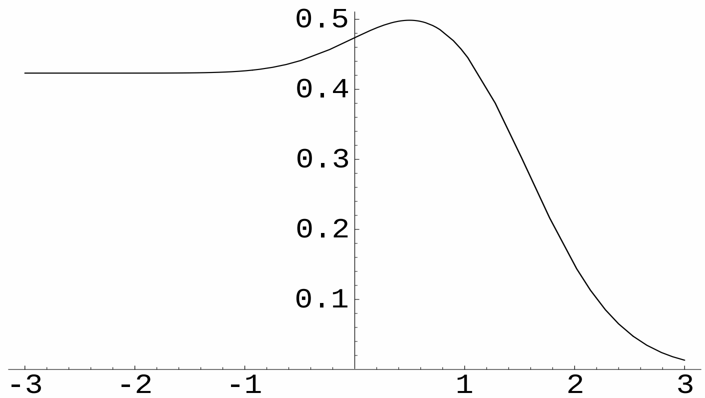

Figure 1: Plot of profit intensity $\rho_N(p')$ in English q-auction for $N=3$

N	$\max_{p'} \rho_N(p')$	$\rho_N(-\infty)$	$\max_{p'} \rho_N(p')/\rho_N(-\infty)$
1	0.27603	0	–
2	0.410091	0.282095	1.45373
3	0.498606	0.423142	1.17834
4	0.564273	0.514688	1.09634
5	0.616195	0.581482	1.0597
6	0.658949	0.633603	1.04
7	0.695165	0.676089	1.02822
8	0.726489	0.7118	1.02064
9	0.754024	0.742507	1.01551
10	0.77854	0.769376	1.01191

Table 1: Profit intensities in units of σ) Gaussian 1th-order statistics

The N-dependence become negligible for large values of N. Tab. 1 presents results gained while using two methods of selection the strategy of fixing withdrawal prices by player -1 and $N \leq 10$. The last column of Tab. 1 contains ratios that do not depend on the dispersion σ of $\eta(q)$. It is obvious that the attractiveness of auction consists in not in abilities of the seller but the rivalry among great number of bidders. The opportunities resulting from growing number of bidders present Fig. 2.

It is easy to notice that for $N \leq 100$ a very good approximation of the profits counted in units σ is given by a logarithmic series if the assumption of

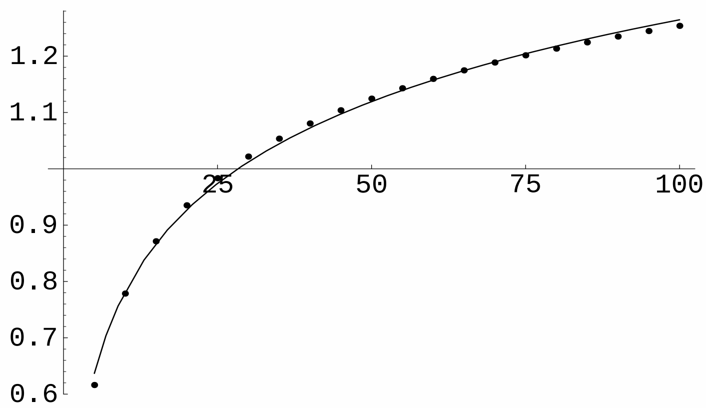

Figure 2: Maximal values of profit intensities in English q-auction for $N \leq 100$ against the curve $0.21 \log N + 0.3$

equality of Gaussian distributions $\eta(q_k)$ is valid. The player -1' profits measured with respect to the mean value of the logarithm of market price of the good being sold must be balanced by the loss winning bidder (modulo the possible brokerage that we neglect). It follows that in case the player -1 does not fix her withdrawal price the intensity of average losses of bidders is equal to $-\frac{2\rho_N(-\infty)}{1+N}$. Therefore the increase in the number of bidders is advantageous to both sides. It is not possible to reduce the number of players by forming linear combinations. Such a characteristics being a direct consequence of the quantum no-delete theorem [12] forbids manipulations on the quantum level and stabilizes the equilibrium gained by pure strategies of bidders acting as anonymous Rest of the World [6].

5 Profit intensities asymptotic behaviour

The present day growing popularity of internet auctions and almost unlimited access to such auction organized by robots raises the of maximal profit intensity in English q-auction with large ($N \gg 100$) number of bidders. In this case the approximation by the function $0.21 \log N + 0.3$ is no longer valid. But fortunately it is possible to find the asymptotic behaviour of the function $\max_{p'}\{\rho_N(p')\}$. To this end it is sufficient to find the asymptotic behaviour of the random variable

$$a_N \mathfrak{q}' + b_N, \tag{8}$$

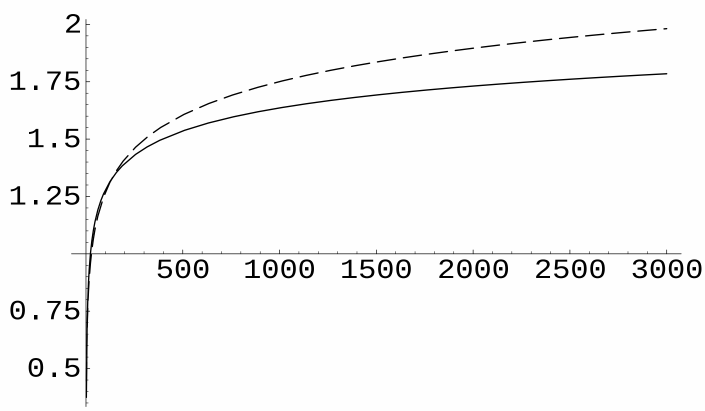

Figure 3: Plots of fuctions $\sqrt{\frac{\ln N}{2}}+\frac{2\gamma-\ln 4\pi-\ln\ln N}{4\sqrt{2\ln N}}$ and $0.21\log N+0.3$ (dashed line)

where the series a_N and b_N are given by

$$a_N := \sqrt{2\ln N} \quad \text{and} \quad b_N := \tfrac{1}{2}(\ln 4\pi + \ln\ln N) - 2\ln N.$$

If $\eta(q)$ is the standard normal distribution then the cumulative distribution function of the random variable (8) being the rescaled logarithm of the price striking the bargain tends to the Gumbel cumulative distribution function (double exponential) [13] [?]

$$P\Big(a_n \mathfrak{q}' + b_n \le x\Big) \xrightarrow{N\to\infty} \mathrm{e}^{-\mathrm{e}^{-x}}.$$

The expectation value of a random variable with probability density $\mathrm{e}^{-\mathrm{e}^{-x}-x}dx$ is equal to the Euler constant $\gamma := \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k} - \ln N \simeq 0.5772$. This after same elementary algebra leads to the asymptotic behaviour of the profit intensity $\max_{p'}\{\rho_N(p')\}$ for $N\to\infty$ of the form

$$\sqrt{\frac{\ln N}{2}} + \frac{2\gamma - \ln 4\pi - \ln\ln N}{4\sqrt{2\ln N}}.$$

The difference between the above function and the previous logarithmic approximation is plotted in Fig. 3.

Details may be found in Cramér's book [10].

6 Bidder's profits

Let us consider in detail the case when the player -1 fixes a unique withdrawal price, the players numbered by $k=1,\ldots,k'-1,k'+1,\ldots,N$ use the same strategies implying Gaussian distribution, but the player k' unlike uses the strategy with fixed withdrawal price $e^{-q'}$ given by the Dirac measure $\delta(q_{k'}-q')dq_{k'}$. Recall [5, 6] that, in the quantum approach, the logarithm of a contingent reselling price of the good in question is indefinite so that $E(\mathfrak{p}_{k'})=0$. Therefore the k'-th player profit intensity is given by

$$\rho_{k'}(q')=\frac{[q'+p'\leq 0]\;q'}{1+\Big(\int\limits_{q'}^{\infty}\eta(q)\;dq\Big)^{1-N}} \tag{9}$$

Fig. 4 presents the shape of the profit intensity function for the three lowest values of N when $p'\to-\infty$ and $\eta(q)$ is the standard normal distribution.

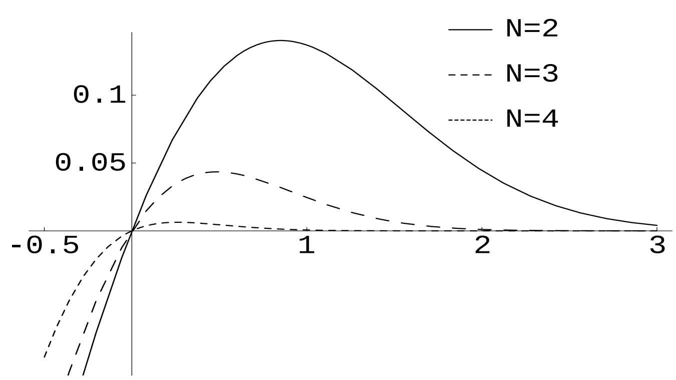

Figure 4: The plot of the bidder's profit intensity as a function of deterministic withdrawal price

For $N=1$ we recover the standard q-bargaining of Ref. [6] and plot we be the strait line given by the equation $\rho_{k'}(q')=\frac{1}{2}q'$. Even if there is only a few active bidders the k'-th player has very limited opportunities of mak ing profits. But if she insists on buying the good she will try to guess such a withdrawal price $\mathrm{e}^{p'}$ to be able to bid the possible highest price q' that would not exceed $-p'$. It is worth to note here that the quantum theory allows to multiply positive profits of a bidder that may be meagre in a single auction. The Pauli exclusion principle does not forbid winning in several auctions if only the players strategy defeated the rivals (it might not result in buying: the sellers withdrawal price might be to high). Immediate teleportation of the state (strategy) [14] makes such quantum market technics possible and effective. The consequences of the fact that strategies cannot be multiplied (undividity of attention) [5] resulting from the no-cloning theorem [15] are not explained by classical models. The possibility of effective using the same strategy at different sites allows to make the profits arbitrary large. This paradox present in classical approaches should incline to research into quantum market games. The no-cloning theorem may also explain our ignorance of our and opponents strategy states: the knowledge would mean cloning.

7 Conditional probabilities in quantum English auctions

The results presented in the previous there paragraphs have to be modified if we suppose that the players joining an auction in the circumstances where the bidders know the prices e^{p_k} at which they may resell the bought good and the price $e^{-q_{\text{-}1}}$ seller paid the good (or the value it presents to him). Adherents of utility theory may that the parameters $q_{\text{-}1}, p_1, \ldots, p_N$ corresponds to the utilities of auctioned good characterizing the appropriate players. So all participants know the value (that may depend on the player) of the good being auctioned. In this case we should substitute the appropriate Wigner functions [5, 6] for the squared absolute values of amplitudes in (3):

$$\begin{aligned} |\langle p|\psi_{\text{-}1}\rangle|^2 &\longrightarrow W_{\text{-}1}(p_{\text{-}1}, q_{\text{-}1}) \\ |\langle q|\psi_k\rangle|^2 &\longrightarrow W_k(p_k, q_k). \end{aligned} \tag{10}$$

So if we take into consideration mixed strategies of participants $\eta_k(p_k, q_k)$ (that is convex linear combinations of Wigner functions) we get

$$dq_k\ \eta_k(p_k, q_k) \prod_{\substack{m=1\\ m\neq k}}^{N} \int_{-\infty}^{\infty} dq_m\ \eta_m(p_m, q_m) \int_{-\infty}^{\infty} dp_{\text{-}1}\ \eta_{\text{-}1}(p_{\text{-}1}, q_{\text{-}1})\ [\ q_k = \min_{n=1,\ldots,N}\{q_n\}\ ]\ [q_k + p_{\text{-}1} \leq 0] \tag{11}$$

instead of the measure (3). The cumulative distribution functions

$$\int_{-\infty}^{p} \eta_k(p_k, q_k = \text{constans})\, dp_k \quad \text{and} \quad \int_{-\infty}^{q} \eta_k(p_k = \text{constans}, q_k)\, dq_k$$

have the natural interpretation of demand and supply curves of the k-th player (if plotted for the common domain $\ln c = p = -q$) [5, 6, 8]. The former analysis of profit intensities is now valid only if $p_1 = \ldots = p_N$ (except for $p_{k'}$) and if all strategies are not giffens (the positiveness of probability measure is supposes in prove of the theorem on maximum of profit intensity). The fascinating class of English q-auctions with giffen strategies requires a separate analysis.

8 Towards a complete theory of quantum auction

The analysis of English q-auction with reversed roles that is bidders are selling is analogous. More interesting is the case when the polarization of the q-auction is changed to $|1\rangle_{-1}|0\rangle_1$. In this case the player -1 reveals her withdrawal price and the player 2 accepts it (and those of the rest of the players) or not. Such an auction is known as the Vickrey's auction (or the second price auction). The winner is obliged to pay the second in decreasing order price from all the bids (and the withdrawal price of the player -1). In the quantum approach English and Vickrey's auctions are only special cases of a phenomenon called q-auction. In the general case both squared absolute values of the amplitudes $|\langle 0_{-1}1_1|0_{-1}1_1\rangle|^2$ and $|\langle 1_{-1}0_1|1_{-1}0_1\rangle|^2$ are non-vanishing so we have consider them with weights corresponding to these probabilities. Such a general q-auction has yet no match on the existing markets. It should be very interesting to analyse the motivation properties of q-auctions eg finding out when the best strategy is the one corresponding to the player's value of the good. The quantum context of the very popular (cf the 1996 Nobel price justification) Vickrey's auction will be analysed in a separate paper.

If we consider only positive definite probability measures then bidder gets the highest profits in Vickrey's auction using strategies with public admission of his valuation of the auctioned good. But it might not be so for giffen strategies because positiveness of measures is supposed in proving

incentive character of Vickrey's auctions [16]. The presence of giffens on real markets might not be so abstract as it seems to be. Captain Robert Giffen who is supposed to find additive measure not being positive definite but present on existing real market in the forties of the XIX century [17] probably got ahead of physicists in observing quantum phenomena. Such departures from the demand low if correctly interpreted do not cause any problem neither for adepts nor for beginners. Employers have probably always thought that work supply as function of payment is scarcely monotonous.
The distinguished by their polarization first and second price auctions have analogues in the Knaster solution to the pragmatic fair division problem that is with compensatory payments for indivisible parts of the property [18]. Such a duality might be found even in election systems that as auctions form procedures of solving fair division problems [19]. It may be that social frustrations caused by election systems should encourage us to discuss such topics.

Acknowledgments. The authors would like to thank dr J. Eisert for stimulating and helpful discussions.

References

[1] L. Lambertini, http://www.spbo.unibo.it/gopher/DSEC/370.pdf. 1

[2] J. Eisert, M. Wilkens, and M. Lewenstein, Phys. Rev. Lett. **83** (1999) 3077.

[3] D. Meyer, Phys. Rev. Lett. **82**, (1999) 1052.

[4] A. Berthiaume, *Quantum Computation* in Complexity Theory Retrospective, vol. 2, ed. A. L. Selman, Springer Verlag(1996). 1

[5] E. W. Piotrowski and J. Sładkowski, submitted to J. Phys. A; quant-ph/0107140. 1, 2, 3, 4, 5, 9, 10, 11

[6] E. W. Piotrowski and J. Sładkowski, submitted to Phys. Lett. A; quant-ph/0104006. 1, 2, 4, 5, 7, 9, 10, 11

[7] R. Penrose, *Shadows of the Mind*, Cambridge University Press, Cambridge (1994). 2

[8] Editor's Note to Comlexity Digest 2001.27(4); http://www.comdig.org. 2, 11

[9] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete Mathematics, Addison-Wesley, Reading, 1994. 2

[10] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, New Jersey (1946). 5, 8

[11] E. W. Piotrowski and J. Sładkowski, submmited to J. Stat.; cond-mat/0102174 5

[12] A. K. Pati and S. L. Braunstein, Nature **404** (2000)164. 7

[13] Encyclopedia of Mathematics, CD-ROM edition, Kluwer Academic Publishers, Dordrecht (1997). 8

[14] C. Bennet et al., Phys. Rev. Lett. **70** (1993) 1895. 10

[15] W. K. Wooters and W. H. Żurek, Nature **299** (1982) 802. 10

[16] P. Klemperer, J. Ec. Surveys **13** 295 (1999). 12

[17] G. J. Stigler, J. Polit. Economy **55** 152 (1947). 12

[18] R. D. Luce and H. Raiffa, *Games and Decisions*, J. Wiley and Sons, New York (1958). 12

[19] S. J. Brams and A. D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution*, Cambridge University Press, (1996). 12