

Hermite polynomials

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In mathematics, the **Hermite polynomials** are a classical orthogonal polynomial sequence.

The polynomials arise in:

- probability, such as the Edgeworth series;
- in combinatorics, as an example of an Appell sequence, obeying the umbral calculus;
- in numerical analysis as Gaussian quadrature;
- in finite element methods as shape functions for beams;
- in physics, where they give rise to the eigenstates of the quantum harmonic oscillator;
- in systems theory in connection with nonlinear operations on Gaussian noise.

Hermite polynomials were defined by Laplace (1810)^[1] though in scarcely recognizable form, and studied in detail by Chebyshev (1859).^[2] Chebyshev's work was overlooked and they were named later after Charles Hermite who wrote on the polynomials in 1864 describing them as new.^[3] They were consequently not new although in later 1865 papers Hermite was the first to define the multidimensional polynomials.

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Definition

There are two different ways of standardizing the Hermite polynomials:

- The "**probabilists' Hermite polynomials**" are given by

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} = \left(x - \frac{d}{dx}\right)^n \cdot 1.$$

- while the "**physicists' Hermite polynomials**" are given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n \cdot 1.$$

These two definitions are not exactly identical; each one is a rescaling of the other,

$$H_n(x) = 2^{\frac{n}{2}} He_n(\sqrt{2}x), \quad He_n(x) = 2^{-\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2}}\right).$$

These are Hermite polynomial sequences of different variances; see the material on variances below.

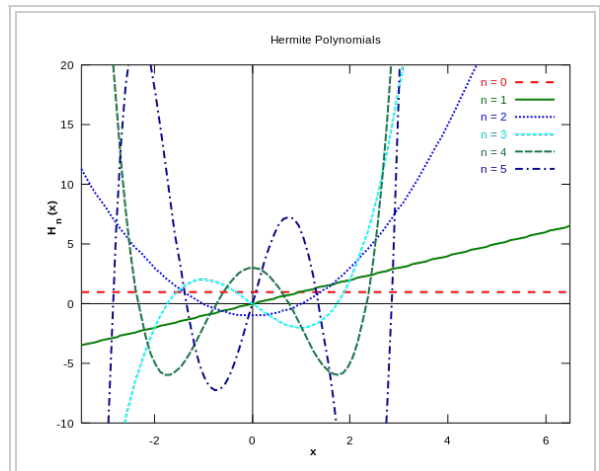
The notation He and H is that used in the standard references Tom H. Koornwinder, Roderick S. C. Wong, and Roelof Koekoek et al. (2010) and Abramowitz & Stegun. The polynomials He_n are sometimes denoted by H_n , especially in probability theory, because

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the probability density function for the normal distribution with expected value 0 and standard deviation 1.

The first eleven probabilists' Hermite polynomials are:

$$\begin{aligned} He_0(x) &= 1 \\ He_1(x) &= x \\ He_2(x) &= x^2 - 1 \\ He_3(x) &= x^3 - 3x \\ He_4(x) &= x^4 - 6x^2 + 3 \\ He_5(x) &= x^5 - 10x^3 + 15x \\ He_6(x) &= x^6 - 15x^4 + 45x^2 - 15 \\ He_7(x) &= x^7 - 21x^5 + 105x^3 - 105x \\ He_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105 \\ He_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x \end{aligned}$$

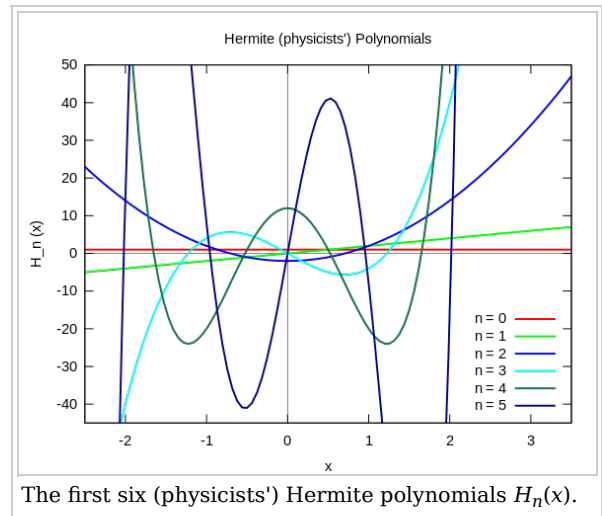


The first six probabilists' Hermite polynomials $He_n(x)$.

$$He_{10}(x) = x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945$$

and the first eleven physicists' Hermite polynomials are:

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \\ H_5(x) &= 32x^5 - 160x^3 + 120x \\ H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120 \\ H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x \\ H_8(x) &= 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680 \end{aligned}$$



$$H_9(x) = 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x$$

$$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$$

Properties

H_n is a polynomial of degree n . The probabilists' version He has leading coefficient 1, while the physicists' version H has leading coefficient 2^n .

Orthogonality

$H_n(x)$ and $He_n(x)$ are n th-degree polynomials for $n = 0, 1, 2, 3, \dots$. These polynomials are orthogonal with respect to the *weight function* (measure)

$$w(x) = e^{-\frac{x^2}{2}} \quad (He)$$

or

$$w(x) = e^{-x^2} \quad (H)$$

i.e., we have

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)w(x)dx = 0, \quad m \neq n.$$

Furthermore,

$$\int_{-\infty}^{\infty} He_m(x)He_n(x)e^{-\frac{x^2}{2}}dx = \sqrt{2\pi n!}\delta_{nm} \quad (\text{probabilists'})$$

or

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = \sqrt{\pi}2^n n! \delta_{nm} \quad (\text{physicists'}).$$

The probabilist polynomials are thus orthogonal with respect to the standard normal probability density function.

Completeness

The Hermite polynomials (probabilist or physicist) form an orthogonal basis of the Hilbert space of functions

satisfying

$$\int_{-\infty}^{\infty} |f(x)|^2 w(x) dx < \infty ,$$

in which the inner product is given by the integral including the Gaussian weight function $w(x)$ defined in the preceding section,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} w(x) dx .$$

An orthogonal basis for $L^2(\mathbf{R}, w(x) dx)$ is a *complete* orthogonal system. For an orthogonal system, *completeness* is equivalent to the fact that the 0 function is the only function $f \in L^2(\mathbf{R}, w(x) dx)$ orthogonal to *all* functions in the system.

Since the linear span of Hermite polynomials is the space of all polynomials, one has to show (in physicist case) that if f satisfies

$$\int_{-\infty}^{\infty} f(x) x^n e^{-x^2} dx = 0$$

for every $n \geq 0$, then $f = 0$.

One possible way to do this is to appreciate that the entire function

$$F(z) = \int_{-\infty}^{\infty} f(x) e^{zx-x^2} dx = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-\infty}^{\infty} f(x) x^n e^{-x^2} dx = 0$$

vanishes identically. The fact then that $F(it) = 0$ for every t real means that the Fourier transform of $f(x) \exp(-x^2)$ is 0, hence f is 0 almost everywhere. Variants of the above completeness proof apply to other weights with exponential decay.

In the Hermite case, it is also possible to prove an explicit identity that implies completeness (see section on the Completeness relation below).

An equivalent formulation of the fact that Hermite polynomials are an orthogonal basis for $L^2(\mathbf{R}, w(x) dx)$ consists in introducing Hermite *functions* (see below), and in saying that the Hermite functions are an orthonormal basis for $L^2(\mathbf{R})$.

Hermite's differential equation

The probabilists' Hermite polynomials are solutions of the differential equation

$$\left(e^{-\frac{x^2}{2}} u' \right)' + \lambda e^{-\frac{1}{2}x^2} u = 0$$

where λ is a constant, with the boundary conditions that u should be polynomially bounded at infinity. With these boundary conditions, the equation has solutions only if λ is a non-negative integer, and up to an overall scaling, the solution is uniquely given by $u(x) = He_{\lambda}(x)$.

Rewriting the differential equation as an eigenvalue problem

$$L[u] = u'' - xu' = -\lambda u ,$$

solutions are the eigenfunctions of the differential operator L . This eigenvalue problem is called the **Hermite equation**, although the term is also used for the closely related equation

$$u'' - 2xu' = -2\lambda u$$

whose solutions are the physicists' Hermite polynomials.

With more general boundary conditions, the Hermite polynomials can be generalized to obtain more general analytic functions $He_\lambda(z)$ for λ a complex index. An explicit formula can be given in terms of a contour integral (Courant & Hilbert 1953).

Recursion relation

The sequence of Hermite polynomials also satisfies the recursion

$$He_{n+1}(x) = xHe_n(x) - He'_n(x). \quad (\text{probabilists'})$$

Individual coefficients are related by the following recursion formula,

$$\begin{aligned} a_{n+1,k} &= a_{n,k-1} - na_{n-1,k} & k > 0 \\ a_{n+1,k} &= -na_{n-1,k} & k = 0 \end{aligned}$$

and $a[0,0]=1$, $a[1,0]=0$, $a[1,1]=1$.

(Assuming $H_n(x) = \sum_{k=0}^n a_{n,k} x^k$)

$$H_{n+1}(x) = 2xH_n(x) - H'_n(x). \quad (\text{physicists'})$$

Individual coefficients are related by the following recursion formula,

$$\begin{aligned} a_{n+1,k} &= 2a_{n,k-1} - (k+1)a_{n,k+1} & k > 0 \\ a_{n+1,k} &= -a_{n,k+1} & k = 0 \end{aligned}$$

and $a[0,0]=1$, $a[1,0]=0$, $a[1,1]=2$.

The Hermite polynomials constitute an Appell sequence, i.e., they are a polynomial sequence satisfying the identity

$$\begin{aligned} He'_n(x) &= nHe_{n-1}(x), & (\text{probabilists'}) \\ H'_n(x) &= 2nH_{n-1}(x). & (\text{physicists'}) \end{aligned}$$

Equivalently, by Taylor expanding,

$$\begin{aligned} He_n(x+y) &= \sum_{k=0}^n \binom{n}{k} x^{n-k} He_k(y) = 2^{-\frac{n}{2}} \cdot \sum_{k=0}^n \binom{n}{k} He_{n-k}(x\sqrt{2}) He_k(y\sqrt{2}) & (\text{probabilists'}) \\ H_n(x+y) &= \sum_{k=0}^n \binom{n}{k} H_k(x)(2y)^{(n-k)} = 2^{-\frac{n}{2}} \cdot \sum_{k=0}^n \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2}). & (\text{physicists'}) \end{aligned}$$

these umbral identities are self-evident and included in the #differential operator representation detailed below,

$$H_n(x) = 2^n e^{-D^2/4} x^n, \quad He_n(x) = e^{-D^2/2} x^n.$$

In consequence, for the m -th derivatives the following relations hold:

$$\begin{aligned} He_n^{(m)}(x) &= \frac{n!}{(n-m)!} \cdot He_{n-m}(x) = m! \cdot \binom{n}{m} \cdot He_{n-m}(x), & (\text{probabilists'}) \\ H_n^{(m)}(x) &= 2^m \cdot \frac{n!}{(n-m)!} \cdot H_{n-m}(x) = 2^m \cdot m! \cdot \binom{n}{m} \cdot H_{n-m}(x). & (\text{physicists'}) \end{aligned}$$

It follows that the Hermite polynomials also satisfy the recurrence relation

$$\begin{aligned} He_{n+1}(x) &= xHe_n(x) - nHe_{n-1}(x), & (\text{probabilists'}) \\ H_{n+1}(x) &= 2xH_n(x) - 2nH_{n-1}(x). & (\text{physicists'}) \end{aligned}$$

These last relations, together with the initial polynomials $H_0(x)$ and $H_1(x)$, can be used in practice to compute

the polynomials quickly.

Turán's inequalities are

$$He_n(x)^2 - He_{n-1}(x)He_{n+1}(x) = (n-1)! \cdot \sum_{i=0}^{n-1} \frac{2^{n-i}}{i!} He_i(x)^2 > 0.$$

Moreover, the following multiplication theorem holds,

$$H_n(\gamma x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma^{n-2i} (\gamma^2 - 1)^i \binom{n}{2i} \frac{(2i)!}{i!} H_{n-2i}(x).$$

Explicit expression

The physicists' Hermite polynomials can be written explicitly as

$$H_n(x) = n! \sum_{\ell=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-\ell}}{(2\ell)! (\frac{n}{2}-\ell)!} (2x)^{2\ell}$$

for even value of n and

$$H_n(x) = n! \sum_{\ell=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-\ell}}{(2\ell+1)! (\frac{n-1}{2}-\ell)!} (2x)^{2\ell+1}$$

for odd values of n .

These two equations may be combined into one using the floor function,

$$H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!} (2x)^{n-2m}.$$

The probabilists' Hermite polynomials He have similar formulas, which may be obtained from these by replacing the power of $2x$ with the corresponding power of $(\sqrt{2})x$, and multiplying the entire sum by $2^{-n/2}$.

$$He_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!} \frac{x^{n-2m}}{2^m}.$$

Inverse Explicit Expression

The inverse of the above explicit expressions, that is, those for monomials in terms of probabilists' Hermite polynomials, He , are

$$x^n = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^m m!(n-2m)!} He_{n-2m}(x).$$

The corresponding expressions for the physicists' Hermite polynomials, H , follow directly by properly scaling this,

$$x^n = \frac{n!}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{m!(n-2m)!} H_{n-2m}(x). \quad [4]$$

Generating function

The Hermite polynomials are given by the exponential generating function

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n=0}^{\infty} He_n(x) \frac{t^n}{n!} \quad (\text{probabilists'})$$

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (\text{physicists'}).$$

This equality is valid for all x, t complex, and can be obtained by writing the Taylor expansion at x of the entire function $z \rightarrow \exp(-z^2)$ (in physicist's case). One can also derive the (physicist's) generating function by using Cauchy's Integral Formula to write the Hermite polynomials as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} \frac{n!}{2\pi i} \oint_{\gamma} \frac{e^{-z^2}}{(z-x)^{n+1}} dz.$$

Using this in the sum

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!},$$

one can evaluate the remaining integral using the calculus of residues and arrive at the desired generating function.

Expected values

If X is a random variable with a normal distribution with standard deviation 1 and expected value μ , then

$$E(He_n(X)) = \mu^n. \quad (\text{probabilists'})$$

The moments of the standard normal may be read off directly from the relation for even indices

$$E(X^{2n}) = (-1)^n He_{2n}(0) = (2n-1)!!$$

where $(2n-1)!!$ is the double factorial. Note that the above expression is a special case of the representation of the probabilists' Hermite polynomials as moments

$$He_n(x) = \int_{-\infty}^{\infty} (x+iy)^n e^{-\frac{y^2}{2}} dy / \sqrt{2\pi}.$$

Asymptotic expansion

Asymptotically, as $n \rightarrow \infty$, the expansion

$$e^{-\frac{x^2}{2}} \cdot H_n(x) \sim \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \cos\left(x\sqrt{2n} - n\frac{\pi}{2}\right) \quad (\text{physicist}^{[5]})$$

holds true. For certain cases concerning a wider range of evaluation, it is necessary to include a factor for changing amplitude

$$e^{-\frac{x^2}{2}} \cdot H_n(x) \sim \frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \cos\left(x\sqrt{2n} - n\frac{\pi}{2}\right) \left(1 - \frac{x^2}{2n}\right)^{-\frac{1}{4}} = \frac{2\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \cos\left(x\sqrt{2n} - n\frac{\pi}{2}\right) \left(1 - \frac{x^2}{2n}\right)^{-\frac{1}{4}}$$

Which, using Stirling's approximation, can be further simplified, in the limit, to

$$e^{-\frac{x^2}{2}} \cdot H_n(x) \sim \left(\frac{2n}{e}\right)^{\frac{n}{2}} \sqrt{2} \cos\left(x\sqrt{2n} - n\frac{\pi}{2}\right) \left(1 - \frac{x^2}{2n}\right)^{-\frac{1}{4}}$$

This expansion is needed to resolve the wave-function of a quantum harmonic oscillator such that it agrees with the classical approximation in the limit of the correspondence principle. A better approximation, which accounts for the variation in frequency is given by

$$e^{-\frac{x^2}{2}} \cdot H_n(x) \sim \left(\frac{2n}{e}\right)^{\frac{n}{2}} \sqrt{2} \cos\left(x\sqrt{2n+1 - \frac{x^2}{3}} - n\frac{\pi}{2}\right) \left(1 - \frac{x^2}{2n}\right)^{-\frac{1}{4}}$$

A finer approximation,^[6] which takes into account the uneven spacing of the zeros near the edges, makes use of the substitution

$$x = \sqrt{2n+1} \cos(\phi), \quad 0 < \epsilon \leq \phi \leq \pi - \epsilon,$$

with which one has the uniform approximation

$$e^{-\frac{x^2}{2}} \cdot H_n(x) = 2^{\frac{n}{2} + \frac{1}{4}} \sqrt{n!} (\pi n)^{-\frac{1}{4}} (\sin \phi)^{-\frac{1}{2}} \cdot \left[\sin\left(\left(\frac{n}{2} + \frac{1}{4}\right) (\sin(2\phi) - 2\phi) + \frac{3\pi}{4}\right) + O(n^{-1}) \right].$$

Similar approximations hold for the monotonic and transition regions. Specifically, if

$$x = \sqrt{2n+1} \cosh(\phi), \quad 0 < \epsilon \leq \phi \leq \omega < \infty,$$

then

$$e^{-\frac{x^2}{2}} \cdot H_n(x) = 2^{\frac{n}{2} - \frac{3}{4}} \sqrt{n!} (\pi n)^{-\frac{1}{4}} (\sinh \phi)^{-\frac{1}{2}} \cdot \exp\left(\left(\frac{n}{2} + \frac{1}{4}\right) (2\phi - \sinh(2\phi))\right) [1 + O(n^{-1})],$$

while for

$$x = \sqrt{2n+1} - 2^{-\frac{1}{2}} 3^{-1/3} n^{-1/6} t$$

with t complex and bounded, then

$$e^{-\frac{x^2}{2}} \cdot H_n(x) = \pi^{\frac{1}{4}} 2^{\frac{n}{2} + \frac{1}{4}} \sqrt{n!} n^{-1/12} [\text{Ai}(-3^{-1/3} t) + O(n^{-2/3})]$$

where $\text{Ai}(\cdot)$ is the Airy function of the first kind.

Special Values

The Hermite polynomials evaluated at zero argument $H_n(0)$ are called Hermite numbers.

$$H_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} 2^{\frac{n}{2}} (n-1)!!, & \text{if } n \text{ is even} \end{cases}$$

which satisfy the recursion relation $H_n(0) = -2(n-1)H_{n-2}(0)$. In terms of the probabilist's polynomials this translates to

$$He_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}} (n-1)!! & \text{if } n \text{ is even.} \end{cases}$$

Relations to other functions

Laguerre polynomials

The Hermite polynomials can be expressed as a special case of the Laguerre polynomials.

$$H_{2n}(x) = (-4)^n n! L_n^{(-\frac{1}{2})}(x^2) = 4^n n! \sum_{i=0}^n (-1)^{n-i} \binom{n-\frac{1}{2}}{n-i} \frac{x^{2i}}{i!} \quad (\text{physicists'})$$

$$H_{2n+1}(x) = 2(-4)^n n! x L_n^{(\frac{1}{2})}(x^2) = 2 \cdot 4^n n! \sum_{i=0}^n (-1)^{n-i} \binom{n + \frac{1}{2}}{n-i} \frac{x^{2i+1}}{i!} \quad (\text{physicists'})$$

Relation to confluent hypergeometric functions

The Hermite polynomials can be expressed as a special case of the parabolic cylinder functions.

$$H_n(x) = 2^n U\left(-\frac{n}{2}, \frac{1}{2}, x^2\right) \quad (\text{physicists'})$$

where $U(a, b, z)$ is Whittaker's confluent hypergeometric function. Similarly,

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} {}_1F_1\left(-n, \frac{1}{2}; x^2\right) \quad (\text{physicists'})$$

$$H_{2n+1}(x) = (-1)^n \frac{(2n+1)!}{n!} 2x {}_1F_1\left(-n, \frac{3}{2}; x^2\right) \quad (\text{physicists'})$$

where ${}_1F_1(a, b; z) = M(a, b; z)$ is Kummer's confluent hypergeometric function.

Differential operator representation

The probabilists' Hermite polynomials satisfy the identity

$$He_n(x) = e^{-\frac{D^2}{2}} x^n,$$

where D represents differentiation with respect to x , and the exponential is interpreted by expanding it as a power series. There are no delicate questions of convergence of this series when it operates on polynomials, since all but finitely many terms vanish.

Since the power series coefficients of the exponential are well known, and higher order derivatives of the monomial x^n can be written down explicitly, this differential operator representation gives rise to a concrete formula for the coefficients of H_n that can be used to quickly compute these polynomials.

Since the formal expression for the Weierstrass transform W is e^{D^2} , we see that the Weierstrass transform of $(\sqrt{2})^n He_n(x/\sqrt{2})$ is x^n . Essentially the Weierstrass transform thus turns a series of Hermite polynomials into a corresponding Maclaurin series.

The existence of some formal power series $g(D)$ with nonzero constant coefficient, such that $He_n(x) = g(D) x^n$, is another equivalent to the statement that these polynomials form an Appell sequence. Since they are an Appell sequence, they are *a fortiori* a Sheffer sequence.

Further information: Weierstrass transform § The inverse

Contour integral representation

From the generating function representation above, we see that the Hermite polynomials have a representation in terms of a contour integral, as

$$He_n(x) = \frac{n!}{2\pi i} \oint \frac{e^{tx - \frac{t^2}{2}}}{t^{n+1}} dt \quad (\text{probabilists'})$$

$$H_n(x) = \frac{n!}{2\pi i} \oint \frac{e^{2tx - t^2}}{t^{n+1}} dt \quad (\text{physicists'})$$

with the contour encircling the origin.

Generalizations

The (probabilists') Hermite polynomials defined above are orthogonal with respect to the standard normal probability distribution, whose density function is

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

which has expected value 0 and variance 1.

Scaling, one may analogously speak of **generalized Hermite polynomials**^[7]

$$He_n^{[\alpha]}(x)$$

of variance α , where α is any positive number. These are then orthogonal with respect to the normal probability distribution whose density function is

$$(2\pi\alpha)^{-\frac{1}{2}} e^{-x^2/(2\alpha)}.$$

They are given by

$$He_n^{[\alpha]}(x) = \alpha^{\frac{n}{2}} He_n \left(\frac{x}{\sqrt{\alpha}} \right) = \left(\frac{\alpha}{2} \right)^{\frac{n}{2}} H_n \left(\frac{x}{\sqrt{2\alpha}} \right) = e^{-\alpha D^2/2} (x^n).$$

Now, if

$$He_n^{[\alpha]}(x) = \sum_{k=0}^n h_{n,k}^{[\alpha]} x^k,$$

then the polynomial sequence whose n th term is

$$\left(He_n^{[\alpha]} \circ He^{[\beta]} \right) (x) \equiv \sum_{k=0}^n h_{n,k}^{[\alpha]} He_k^{[\beta]}(x)$$

is called the **umbral composition** of the two polynomial sequences. It can be shown to satisfy the identities

$$\left(He_n^{[\alpha]} \circ He^{[\beta]} \right) (x) = He_n^{[\alpha+\beta]}(x)$$

and

$$He_n^{[\alpha+\beta]}(x+y) = \sum_{k=0}^n \binom{n}{k} He_k^{[\alpha]}(x) He_{n-k}^{[\beta]}(y).$$

The last identity is expressed by saying that this parameterized family of polynomial sequences is a **cross-sequence**. (See the above section on Appel sequences and on the #Differential operator representation, which leads to a ready derivation of it. This binomial type identity, for $\alpha = \beta = 1/2$, has already been encountered in the above section on #Recursion relations.)

"Negative variance"

Since polynomial sequences form a group under the operation of umbral composition, one may denote by

$$He_n^{[-\alpha]}(x)$$

the sequence that is inverse to the one similarly denoted but without the minus sign, and thus speak of Hermite polynomials of negative variance. For $\alpha > 0$, the coefficients of $He_n^{[-\alpha]}(x)$ are just the absolute values of the corresponding coefficients of $He_n^{[\alpha]}(x)$.

These arise as moments of normal probability distributions: The n -th moment of the normal distribution with expected value μ and variance σ^2 is

$$E(X^n) = He_n^{[-\sigma^2]}(\mu)$$

where X is a random variable with the specified normal distribution. A special case of the cross-sequence identity then says that

$$\sum_{k=0}^n \binom{n}{k} He_k^{[\alpha]}(x) He_{n-k}^{[-\alpha]}(y) = He_n^{[0]}(x+y) = (x+y)^n.$$

Applications

Hermite functions

One can define the **Hermite functions** from the physicists' polynomials:

$$\psi_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}.$$

Since these functions contain the square root of the weight function, and have been scaled appropriately, they are orthonormal:

$$\int_{-\infty}^{\infty} \psi_n(x) \psi_m(x) dx = \delta_{nm}$$

and form an orthonormal basis of $L^2(\mathbf{R})$. This fact is equivalent to the corresponding statement for Hermite polynomials (see above).

The Hermite functions are closely related to the Whittaker function (Whittaker and Watson, 1962) $D_n(z)$,

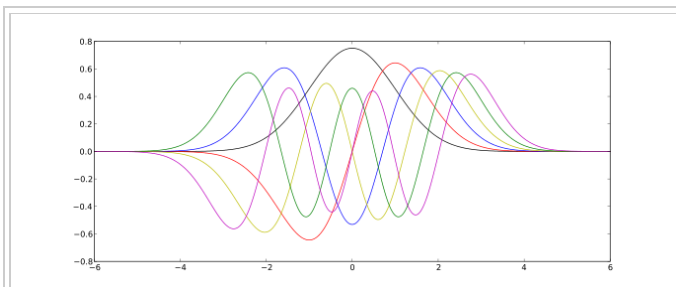
$$D_n(z) = (n! \sqrt{\pi})^{\frac{1}{2}} \psi_n(z/\sqrt{2}) = \pi^{-\frac{1}{4}} \sqrt{2} e^{z^2/4} \frac{d^n}{dz^n} e^{-z^2},$$

and thereby to other parabolic cylinder functions.

The Hermite functions satisfy the differential equation,

$$\psi_n''(x) + (2n + 1 - x^2)\psi_n(x) = 0.$$

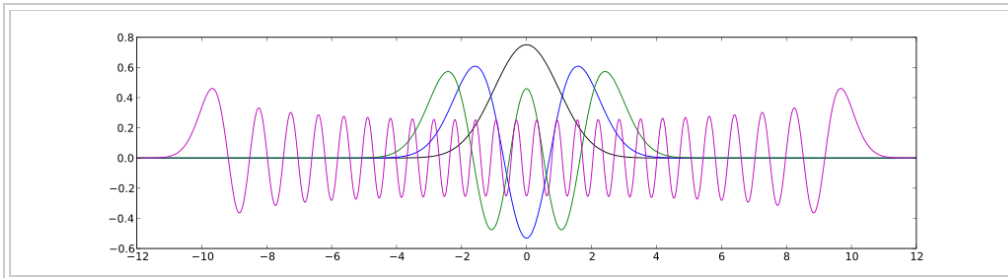
This equation is equivalent to the Schrödinger equation for a harmonic oscillator in quantum mechanics, so these functions are the eigenfunctions.



Hermite functions 0 (black), 1 (red), 2 (blue), 3 (yellow), 4 (green), and 5 (magenta).

$$\begin{aligned} \psi_0(x) &= \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2} \\ \psi_1(x) &= \sqrt{2} \pi^{-\frac{1}{4}} x e^{-\frac{1}{2}x^2} \\ \psi_2(x) &= (\sqrt{2} \pi^{\frac{1}{4}})^{-1} (2x^2 - 1) e^{-\frac{1}{2}x^2} \end{aligned}$$

$$\begin{aligned}\psi_3(x) &= (\sqrt{3} \pi^{\frac{1}{4}})^{-1} (2x^3 - 3x) e^{-\frac{1}{2}x^2} \\ \psi_4(x) &= (2\sqrt{6} \pi^{\frac{1}{4}})^{-1} (4x^4 - 12x^2 + 3) e^{-\frac{1}{2}x^2} \\ \psi_5(x) &= (2\sqrt{15} \pi^{\frac{1}{4}})^{-1} (4x^5 - 20x^3 + 15x) e^{-\frac{1}{2}x^2}\end{aligned}$$



Hermite functions 0 (black), 2 (blue), 4 (green), and 50 (magenta).

Recursion relation

Following recursion relations of Hermite polynomials, the Hermite functions obey

$$\psi'_n(x) = \sqrt{\frac{n}{2}} \psi_{n-1}(x) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(x),$$

as well as

$$x \psi_n(x) = \sqrt{\frac{n}{2}} \psi_{n-1}(x) + \sqrt{\frac{n+1}{2}} \psi_{n+1}(x).$$

Extending the first relation to the arbitrary m -th derivatives for any positive integer m leads to

$$\psi_n^{(m)}(x) = \sum_{k=0}^m \binom{m}{k} (-1)^k 2^{(m-k)/2} \sqrt{\frac{n!}{(n-m+k)!}} \cdot \psi_{n-m+k}(x) \cdot He_k(x).$$

This formula can be used in connection with the recurrence relations for He_n and ψ_n to calculate any derivative of the Hermite functions efficiently.

Cramér's inequality

The Hermite functions satisfy the following bound due to Harald Cramér^{[8][9]}

$$|\psi_n(x)| \leq K \pi^{-\frac{1}{4}}$$

for x real, where the constant K is less than 1.086435.

Hermite functions as eigenfunctions of the Fourier transform

The Hermite functions $\psi_n(x)$ are a set of eigenfunctions of the continuous Fourier transform. To see this, take the physicist's version of the generating function and multiply by $\exp(-x^2/2)$. This gives

$$\exp(-x^2/2 + 2xt - t^2) = \sum_{n=0}^{\infty} \exp(-x^2/2) H_n(x) \frac{t^n}{n!}.$$

Choosing the unitary representation of the Fourier transform, the Fourier transform of the left hand side is given by

$$\begin{aligned}\mathcal{F}\left\{\exp\left(-\frac{x^2}{2}+2xt-t^2\right)\right\}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ixk)\exp(-x^2/2+2xt-t^2)\,dx \\ &= \exp(-k^2/2-2kit+t^2) \\ &= \sum_{n=0}^{\infty}\exp\left(-\frac{k^2}{2}\right)H_n(k)\frac{(-it)^n}{n!}\end{aligned}$$

The Fourier transform of the right hand side is given by

$$\mathcal{F}\left\{\sum_{n=0}^{\infty}e^{-\frac{x^2}{2}}H_n(x)\frac{t^n}{n!}\right\}=\sum_{n=0}^{\infty}\mathcal{F}\left\{e^{-\frac{x^2}{2}}H_n(x)\right\}\frac{t^n}{n!}.$$

Equating like powers of t in the transformed versions of the left- and right-hand sides finally yields

$$\mathcal{F}\left\{e^{-\frac{x^2}{2}}H_n(x)\right\}=(-i)^ne^{-\frac{k^2}{2}}H_n(k).$$

The Hermite functions $\psi_n(x)$ are thus an orthonormal basis of $L^2(\mathbb{R})$ which *diagonalizes the Fourier transform operator*.

In this case, we chose the unitary version of the Fourier transform, so the eigenvalues are $(-i)^n$. The ensuing resolution of the identity then serves to define powers, including fractional ones, of the Fourier transform, to wit a Fractional Fourier transform generalization, in effect a Mehler kernel.

Wigner distributions of Hermite functions

The Wigner distribution function of the n -th order Hermite function is related to the n -th order Laguerre polynomial. The Laguerre polynomials are

$$L_n(x):=\sum_{k=0}^n\binom{n}{k}\frac{(-1)^k}{k!}x^k,$$

leading to the oscillator Laguerre functions,

$$l_n(x):=e^{-\frac{x}{2}}L_n(x).$$

For all natural integers n , it is straightforward to see^[10] that

$$W_{\psi_n}(t,f)=(-1)^nl_n\left(4\pi(t^2+f^2)\right),$$

where the Wigner distribution of a function $x\in L^2(\mathbb{R},\mathbb{C})$ is defined as

$$W_x(t,f)=\int_{-\infty}^{\infty}x(t+\tau/2)x(t-\tau/2)^*e^{-2\pi i\tau f}\,d\tau.$$

This is a fundamental result for the quantum harmonic oscillator discovered by Hip Groenewold in 1946 in his PhD thesis.^[11] It is the standard paradigm of quantum mechanics in phase space.

There are further relations between the two families of polynomials.

Combinatorial interpretation of coefficients

In the Hermite polynomial $He_n(x)$ of variance 1, the absolute value of the coefficient of x^k is the number of (unordered) partitions of an n -member set into k singletons and $(n-k)/2$ (unordered) pairs. The sum of the absolute values of the coefficients gives the total number of partitions into singletons and pairs, the so-called telephone numbers

1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ... (sequence A000085 in OEIS).

These numbers may also be expressed as a special value of the Hermite polynomials^[12]

$$T(n) = \frac{He_n(i)}{i^n}.$$

Completeness relation

The Christoffel-Darboux formula for Hermite polynomials reads

$$\sum_{k=0}^n \frac{H_k(x)H_k(y)}{k!2^k} = \frac{1}{n!2^{n+1}} \frac{H_n(y)H_{n+1}(x) - H_n(x)H_{n+1}(y)}{x - y}.$$

Moreover, the following completeness identity for the above Hermite functions holds in the sense of distributions

$$\sum_{n=0}^{\infty} \psi_n(x)\psi_n(y) = \delta(x - y),$$

where δ is the Dirac delta function, ψ_n the Hermite functions, and $\delta(x - y)$ represents the Lebesgue measure on the line $y = x$ in \mathbb{R}^2 , normalized so that its projection on the horizontal axis is the usual Lebesgue measure.

This distributional identity follows (N.Wiener^[13]) by taking $u \rightarrow 1$ in Mehler's formula, valid when $-1 < u < 1$,

$$E(x, y; u) := \sum_{n=0}^{\infty} u^n \psi_n(x)\psi_n(y) = \frac{1}{\sqrt{\pi(1-u^2)}} \exp\left(-\frac{1-u}{1+u} \frac{(x+y)^2}{4} - \frac{1+u}{1-u} \frac{(x-y)^2}{4}\right),$$

which is often stated equivalently as a separable kernel,^{[14][15]}

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} \left(\frac{u}{2}\right)^n = \frac{1}{\sqrt{1-u^2}} e^{\frac{2u}{1+u}xy - \frac{u^2}{1-u^2}(x-y)^2}.$$

The function $(x, y) \rightarrow E(x, y; u)$ is the bivariate Gaussian probability density on \mathbb{R}^2 which is, when u is close to 1, very concentrated around the line $y = x$, and very spread out on that line. It follows that

$$\left\langle \sum_{n=0}^{\infty} u^n \langle f, \psi_n \rangle \psi_n, g \right\rangle = \iint E(x, y; u) f(x) \overline{g(y)} dx dy \rightarrow \int f(x) \overline{g(x)} dx = \langle f, g \rangle,$$

when f, g are continuous and compactly supported.

This yields that f can be expressed in Hermite functions, as the sum of a series of vectors in $L^2(\mathbb{R})$, namely

$$f = \sum_{n=0}^{\infty} \langle f, \psi_n \rangle \psi_n.$$

In order to prove the above equality for $E(x, y; u)$, the Fourier transform of Gaussian functions is used repeatedly,

$$\rho\sqrt{\pi}e^{-\frac{\rho^2 x^2}{4}} = \int e^{isx - s^2/\rho^2} ds, \quad \rho > 0.$$

The Hermite polynomial is then represented as

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(\frac{1}{2\sqrt{\pi}} \int e^{isx - \frac{s^2}{4}} ds \right) = (-1)^n e^{x^2} \frac{1}{2\sqrt{\pi}} \int (is)^n e^{isx - \frac{s^2}{4}} ds.$$

With this representation for $H_n(x)$ and $H_n(y)$, it is evident that

$$\begin{aligned}
 E(x, y; u) &= \sum_{n=0}^{\infty} \frac{u^n}{2^n n! \sqrt{\pi}} H_n(x) H_n(y) e^{-\frac{x^2+y^2}{2}} \\
 &= \frac{e^{\frac{x^2+y^2}{2}}}{4\pi\sqrt{\pi}} \iint \left(\sum_{n=0}^{\infty} \frac{1}{2^n n!} (-ust)^n \right) e^{isx+ity-\frac{s^2}{4}-\frac{t^2}{4}} ds dt \\
 &= \frac{e^{\frac{x^2+y^2}{2}}}{4\pi\sqrt{\pi}} \iint e^{-ust/2} e^{isx+ity-\frac{s^2}{4}-\frac{t^2}{4}} ds dt
 \end{aligned}$$

and this yields the desired resolution of the identity result, using again the Fourier transform of Gaussian kernels under the substitution

$$s = \frac{\sigma + \tau}{\sqrt{2}}, \quad t = \frac{\sigma - \tau}{\sqrt{2}}.$$

See also

- Mehler kernel
- Turán's inequalities
- Parabolic cylinder function

Notes

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