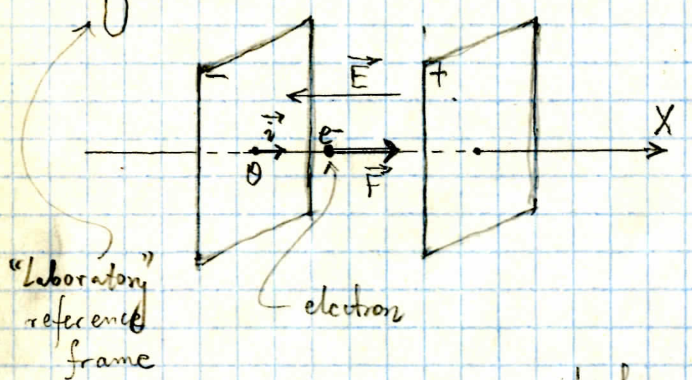
Relativistic dynamics

Motion of a point charge in uniform and time-independent electric field: a vacuum parallel plate capacitor.



In Newtonian mechanics:

$$m\vec{a} = \vec{F} = q\vec{E} = -e\vec{E} \quad (e>0),$$

$\vec{E}$ – vector of the electric field,

m – mass of electron, $-e$ – charge of electron.

Let us assume that electron moves along the x axis,

$$\vec{a} = a_x\vec{i}, \quad \vec{E} = -E\vec{i} \quad (E = |\vec{E}| > 0),$$

$$m a_x \vec{i} = -e(-E\vec{i}) = +eE\vec{i} \Rightarrow m a_x = eE,$$

thus $a_x = \dfrac{eE}{m} = \text{const}$, electron's acceleration is constant,

therefore $v_x(t) = v_{0x} + a_x t \quad [v_{0x} = v_x(t=0)]$,

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \quad [x_0 = x(t=0)].$$

For $t \to \infty$, $v_x(t) \to \infty$, what contradicts special relativity: one has to modify the 2nd Newton's law of dynamics.

Let at the moment of time t the instantaneous velocity of the electron with respect to laboratory will be $v_x(t)$. Let us introduce an IRF U' moving with velocity $v_x(t)$ with respect to laboratory (this IRF is comoving with the electron at time t). Obviously, at time t the electron is at rest in U'.

For a very short time interval $\Delta t'$ (measured in U' from the moment of time $t'=0$ corresponding to time t in U) one can employ Newtonian 2nd law of dynamics: from $t'=0$ to $t'=\Delta t'$ the electron will speed up from zero velocity to the velocity

$$\Delta v'_{x'} = a'_{x'}\Delta t' + \underbrace{O((\Delta t')^2)}_{\text{higher-order relativistic correction}},$$

where $a'_{x'} = \dfrac{eE'}{m}$, E' is the value of the electric field in U'.

During the time $\Delta t'$ the x' coordinate of the electron will change by quantity

$$\Delta x' = \frac{1}{2}a'_{x'}(\Delta t')^2 + O((\Delta t')^3).$$

The change $\Delta v'_{x'}$ of the electron's velocity in U' is related with the change Δv_x of its velocity with respect to U by relativistic law of composition of velocities:

$$\underbrace{\Delta v_x + v_x(t)}_{\substack{\text{velocity at time } t+\Delta t \\ \text{(with respect to } U)}} = \frac{v_x(t) + \Delta v'_{x'}}{1 + \frac{v_x(t)}{c^2}\Delta v'_{x'}} \Longrightarrow \Delta v_x = \frac{1 - \frac{(v_x(t))^2}{c^2}}{1 + \frac{v_x(t)}{c^2}\Delta v'_{x'}}\Delta v'_{x'}$$

$$= \left(1 - \frac{(v_x(t))^2}{c^2}\right)\Delta v'_{x'} + O((\Delta v'_{x'})^2),$$

but $\Delta v'_{x'} = O(\Delta t')$, so

$$\Delta v_x = \left(1 - \frac{(v_x(t))^2}{c^2}\right)\Delta v'_{x'} + O((\Delta t')^2).$$

Time interval $\Delta t'$ in U' corresponds to time interval Δt in U; from Lorentz transformation

$$\Delta t = t_2 - t_1 = \gamma\left(t'_2 + \frac{u}{c^2}x'_2\right) - \gamma\left(t'_1 + \frac{u}{c^2}x'_1\right)$$

$$= \gamma\left[(t'_2 - t'_1) + \frac{u}{c^2}(x'_2 - x'_1)\right] = \gamma\left(\Delta t' + \frac{u}{c^2}\Delta x'\right),$$

in our case $u = v_x(t)$ and $\Delta x' = \frac{1}{2}a'_{x'}(\Delta t')^2 + O((\Delta t')^3)$,

therefore $$\Delta t = \frac{\Delta t' + \frac{v_x(t)}{c^2}\Delta x'}{\sqrt{1 - [v_x(t)]^2/c^2}} = \frac{\Delta t'}{\sqrt{1 - [v_x(t)]^2/c^2}} + O((\Delta t')^2).$$

Let us observe that if $\Delta t' \to 0$, then also $\Delta t \to 0$ (and vice versa).

Acceleration of the electron in U:

$$a_x = \lim_{\Delta t \to 0}\frac{\Delta v_x}{\Delta t} = \lim_{\Delta t' \to 0}\frac{\left(1 - [v_x(t)]^2/c^2\right)\Delta v'_{x'} + O((\Delta t')^2)}{\frac{\Delta t'}{\sqrt{1 - [v_x(t)]^2/c^2}} + O((\Delta t')^2)}$$

$$= \lim_{\Delta t' \to 0}\frac{\left(1 - [v_x(t)]^2/c^2\right)\frac{\Delta v'_{x'}}{\Delta t'} + O(\Delta t')}{\frac{1}{\sqrt{1 - [v_x(t)]^2/c^2}} + O(\Delta t')}$$

$$= \lim_{\Delta t' \to 0}\left\{\left(1 - \frac{[v_x(t)]^2}{c^2}\right)^{3/2}\frac{\Delta v'_{x'}}{\Delta t'} + O(\Delta t')\right\}$$

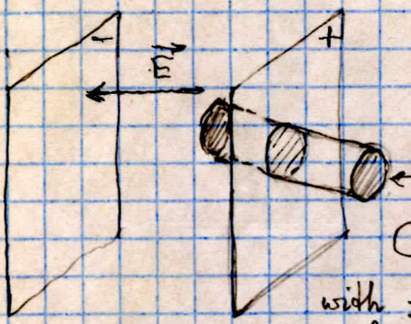
$$= \left(1 - \frac{[v_x(t)]^2}{c^2}\right)^{3/2} a'_{x'} = \frac{e}{m}E'\left(1 - \frac{[v_x(t)]^2}{c^2}\right)^{3/2},$$

where we have employed the fact, that for small x ($|x| \ll 1$)

$$\frac{a + x}{b + x} = \frac{a}{b} + \frac{b - a}{b^2}x + \dots .$$

✓

We show now that $E' = E$. From Gauss's law applied in the IRF U:



the electric flux through the cylinder

$$E\,\Delta S = \frac{1}{\varepsilon_0}\Delta Q \implies E = \frac{1}{\varepsilon_0}\frac{\Delta Q}{\Delta S} = \frac{\sigma}{\varepsilon_0},$$

where σ is the surface charge density (ε_0 is the permittivity of free space).

a cylinder with flat ends of area ΔS and axis perpendicular to the charged plane, the cylinder is at rest in IRF U

In IRF U′ one consider a cylinder which is at rest with respect to U′, so the charged plane has a velocity $-v_x(t)$ with respect to this cylinder. But the Gauss's law can be applied regardless the charges inside the Gaussian surface are at rest or are in motion. Therefore Gauss law applied in the IRF U′ gives:

$$E'\,\Delta S = \frac{1}{\varepsilon_0}\Delta Q \implies E' = \frac{1}{\varepsilon_0}\frac{\Delta Q}{\Delta S} = E.$$

Finally, the acceleration of the electron in U reads

$$a_x(t) = \frac{eE}{m}\left(1 - \frac{[v_x(t)]^2}{c^2}\right)^{3/2}.$$

We see that when $v_x(t) \ll c$, the acceleration $a_x(t) \cong \frac{eE}{m} = \text{const}$, whereas for $v_x(t) \to c$, the acceleration $a_x(t) \to 0$:

this is the reason why the speed of light is never achieved by the electron, even if time of speeding up of electron is infinitely long.

Equation of motion of the electron can be written as

$$\frac{m\,a_x(t)}{\left(1 - \frac{(v_x(t))^2}{c^2}\right)^{3/2}} = eE.$$

By direct computation one checks that $\frac{d}{dt}\frac{m v_x(t)}{\sqrt{1-(v_x(t))^2/c^2}} = \frac{m\,a_x(t)}{\left(1-(v_x(t))^2/c^2\right)^{3/2}}$,

Therefore the relativistic equation of motion of the electron in the electric field (with velocity parallel to the vector of the electric field) has the form

$$\frac{d}{dt}\frac{m v_x}{\sqrt{1 - \frac{v_x^2}{c^2}}} = eE. \qquad (*)$$

√

It can be shown that the most general form of the equation of motion of a point particle with electric charge q moving in arbitrary electromagnetic field (with electric field $\vec{E}$ and magnetic field $\vec{B}$) reads

$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = \underbrace{q\vec{E}}_{\text{electric force}} + \underbrace{q\vec{v}\times\vec{B}}_{\text{magnetic force}}. \qquad (**)$$

Equation (**) e.g. explains operation of modern particle accelerators such as the LHC accelerator located in CERN.
The LHC is to speed up protons to velocities for which the Lorentz factor $\gamma \cong 7.500$ (this corresponds to velocity $v \cong 0.999\,999\,991c$).

Let us assume that $v_x(t=0)=0$ and $x(t=0)=0$.
Then, integrating Eq. (*) two times, one gets

$v_x(t) = \dfrac{a_0 t}{\sqrt{1+\frac{a_0^2t^2}{c^2}}}$, where $a_0 := \dfrac{eE}{m}$ is the acceleration of the particle in momentarily comoving reference frame,

$$x(t) = \frac{c^2}{a_0}\left(\sqrt{1+\frac{a_0^2t^2}{c^2}}-1\right).$$

We see that $\quad v_x(t) \xrightarrow[t\to\infty]{} \dfrac{a_0 t}{\sqrt{\frac{a_0^2t^2}{c^2}}} = \dfrac{a_0 t}{\frac{a_0 t}{c}} = c.$

For times t so small that $a_0 t \ll c$:

$$v_x(t) \cong a_0 t,$$
$$x(t) \cong \frac{c^2}{a_0}\left(1+\frac{a_0^2t^2}{2c^2}-1\right) = \frac{1}{2}a_0 t^2.$$

The expression $m\vec{v}/\sqrt{1-v^2/c^2}$ which appears on the left-hand side of Eq. (**) corresponds to the Newtonian momentum $m\vec{v}$ of the particle in the non-relativistic version of Eq. (**). It defines the momentum of the particle in special relativity:

$$\vec{p} := \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}.$$

Experiments confirm that the momentum such defined is a quantity which is conserved during collisions of particles.

The Lorentz factor $\gamma = 1/\sqrt{1-v^2/c^2}$ is a function of the modulus v of the particle's velocity $\vec{v}$: $\gamma = \gamma(v)$.
For small velocities ($v \ll c$, that is $v/c \ll 1$) one can expand $\gamma(v)$ in the Taylor series with respect to v around $v=0$:

$$\gamma(v) = \gamma(0) + \gamma'(0)v + \frac{1}{2}\gamma''(0)v^2 + \frac{1}{3!}\gamma'''(0)v^3 + \dots$$
$$= 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + O(v^6);$$

making use of this expansion one gets

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}} = m\vec{v}\Big[\underbrace{1}_{\text{Newtonian momentum}} + \underbrace{\frac{1}{2}\left(\frac{v}{c}\right)^2}_{\text{1st relativistic correction}} + \underbrace{\frac{3}{8}\left(\frac{v}{c}\right)^4 + O(v^6)}_{\text{higher-order relativistic correction}}\Big].$$

Quite often a notion of <u>relativistic mass</u> is introduced:

$$m_r := \frac{m}{\sqrt{1-v^2/c^2}},$$

then the momentum of a relativistic particle can be written as $\vec{p} = m_r\vec{v}$.

When $\vec{v} = \vec{0}$, $m_r = m$, this is why m is called the <u>rest mass</u> of the particle.

Often the rest mass is denoted by m_0, whereas m is reserved for relativistic mass.

Relativistic kinetic energy

Kinetic energy of a particle is defined by two requirements:

(i) kinetic energy of a particle at rest is zero;

(ii) change of kinetic energy in time interval $\langle t_1; t_2\rangle$ is equal to the work done by the net force acting on the particle in that interval:

$$\Delta E_k = E_k(t_2) - E_k(t_1) = \int_{t_1}^{t_2} \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F}\cdot \frac{d\vec{r}}{dt}\, dt = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\, dt .$$

Let us consider an infinitesimal time interval $\langle t; t+\Delta t\rangle$, then

$$\Delta E_k = \int_{t}^{t+\Delta t} \vec{F}\cdot\vec{v}\, dt \cong \vec{F}\cdot\vec{v}\,\Delta t \implies \frac{\Delta E_k}{\Delta t} \cong \vec{F}\cdot\vec{v},$$

taking the limit $\Delta t \to 0$: $\quad \dfrac{dE_k}{dt} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta E_k}{\Delta t} = \vec{F}\cdot\vec{v}.$

Relativistic equation of motion can be written as

$$\vec{F} = \frac{d\vec{p}}{dt}, \quad \text{where} \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}},$$

therefore $\quad \dfrac{dE_k}{dt} = \vec{v}\cdot\dfrac{d\vec{p}}{dt} = \vec{v}\cdot\dfrac{d}{dt}\dfrac{m\vec{v}}{\sqrt{1-v^2/c^2}}.$

By direct computation one checks that $\quad \vec{v}\cdot\dfrac{d}{dt}\dfrac{m\vec{v}}{\sqrt{1-v^2/c^2}} = \dfrac{d}{dt}\dfrac{mc^2}{\sqrt{1-v^2/c^2}},$

hence $\quad 0 = \dfrac{dE_k}{dt} - \dfrac{d}{dt}\dfrac{mc^2}{\sqrt{1-v^2/c^2}} = \dfrac{d}{dt}\left(E_k - \dfrac{mc^2}{\sqrt{1-v^2/c^2}}\right),$

thus $\quad E_k = \dfrac{mc^2}{\sqrt{1-v^2/c^2}} + C, \quad$ where $C = \text{const}.$

But $E_k = 0$ when $\vec{v} = \vec{0}$, therefore $C = -mc^2$ and finally

$$\underline{\underline{E_k = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2.}}$$

E_k is a function of the speed v of the particle, $E_k = E_k(v)$. For small velocities one can expand $E_k(v)$ in Taylor series around $v=0$:

$$\begin{aligned} E_k(v) &= \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^4} + O(v^6) \\ &= \frac{1}{2}mv^2\left[1 + \frac{3}{4}\left(\frac{v}{c}\right)^2 + O(v^4)\right]. \end{aligned}$$

The principle of conservation of energy

The total energy of a free body is the sum of its kinetic energy and internal energy. We postulate that in special relativity the principle of conservation of total energy is valid.

Let us consider the collision of r particles after which s (maybe new) particles appear; the principle of conservation of total energy says that

$$\sum_{a=1}^{r} \left(E_{k,a} + w_a \right) = \sum_{b=1}^{s} \left(\tilde{E}_{k,b} + \tilde{w}_b \right).$$

kinetic energy of a-th particle before collision

internal energy of a-th particle before collision

kinetic energy and internal energy of b-th particle after collision

To simplify computations let us consider situation in which all particles before and after collision are moving along the axis x of some IRF U.
The a-th particle has velocity $\vec{v}_a$ with respect to U,
with respect to another IRF U' it will have velocity $\vec{v}'_a$,

$$\vec{v}_a = v_{ax}\vec{i}, \quad \vec{v}'_a = v'_{ax'}\vec{i}', \quad v_{ax} = \frac{v'_{ax'} + u}{1 + \frac{u v'_{ax'}}{c^2}}.$$

unit vector of the axis x in U

unit vector of the axis x' in U'

For shortness let us replace v_{ax} by v_a and $v'_{ax'}$ by v'_a, then $v_a = \dfrac{v'_a + u}{1 + \frac{u v'_a}{c^2}}$.

Let us introduce notation $E_a := \dfrac{m_a c^2}{\sqrt{1 - v_a^2/c^2}}$, then

$$E_{k,a} = \frac{m_a c^2}{\sqrt{1 - v_a^2/c^2}} - m_a c^2 = E_a - m_a c^2.$$

One can show that

$$E_a = \frac{m_a c^2}{\sqrt{1 - v_a^2/c^2}} = \frac{1}{\sqrt{1 - u^2/c^2}} \left(\frac{m_a c^2}{\sqrt{1 - v_a'^2/c^2}} + u \frac{m_a v'_a}{\sqrt{1 - v_a'^2/c^2}} \right) = \frac{1}{\sqrt{1 - u^2/c^2}} \left(E'_a + u p'_a \right). \quad (1)$$

x' component of the momentum of the a-th particle

We write the principle of conservation of total energy in both IRFs U and U':

$$\text{in } U: \quad \sum_{a=1}^{r} \left(E_a + w_a - m_a c^2 \right) = \sum_{b=1}^{s} \left(\tilde{E}_b + \tilde{w}_b - \tilde{m}_b c^2 \right), \quad (2)$$

$$\text{in } U': \quad \sum_{a=1}^{r} \left(E'_a + w_a - m_a c^2 \right) = \sum_{b=1}^{s} \left(\tilde{E}'_b + \tilde{w}_b - \tilde{m}_b c^2 \right). \quad (3)$$

In Eq. (2), making use of Eq. (1), we express E_a by E'_a, p'_a and $\tilde{E}_b$ by $\tilde{E}'_b, \tilde{p}'_b$. Such obtained equation we simplify using Eq. (3). Finally we obtain

$$\sum_{a=1}^{r} p'_a - \sum_{b=1}^{s} p'_b = \frac{\sqrt{1-u^2/c^2}-1}{u}\left(\sum_{b=1}^{s}(\tilde{w}_b - \tilde{m}_b c^2) - \sum_{a=1}^{r}(w_a - m_a c^2)\right).$$

This equality, for given fixed momenta p'_a and p'_b, has to be fulfilled for any value of the relative velocity u. This is possible only when both sides of this equality are simultaneously equal to zero:

$$\sum_{a=1}^{r} p'_a = \sum_{b=1}^{s} p'_b$$ – conservation of momentum,

$$\sum_{a=1}^{r}(w_a - m_a c^2) = \sum_{b=1}^{s}(\tilde{w}_b - \tilde{m}_b c^2)$$ – we substitute this into Eq. (2); we get

$$\sum_{a=1}^{r} E_a = \sum_{b=1}^{s} \tilde{E}_b.$$ This equation can be rewritten as

$$\sum_{a=1}^{r}(E_{k,a} + m_a c^2) = \sum_{b=1}^{s}(\tilde{E}_{k,b} + \tilde{m}_b c^2),$$ or still in another form:

$$\underline{\Delta E_k = -(\Delta m)c^2}, \quad \text{where} \quad \Delta E_k := \sum_{b=1}^{s} \tilde{E}_{k,b} - \sum_{a=1}^{r} E_{k,a},$$

$$\Delta m := \sum_{b=1}^{s} \tilde{m}_b - \sum_{a=1}^{r} m_a.$$

Gain (loss) of total kinetic energy of colliding particles is related to loss (gain) of total mass of the particles multiplied by c^2.
It implies that <u>internal energy</u> of the particle is given by

$W_a = m_a c^2$, while <u>total energy</u> is equal to $E_a = \dfrac{m_a c^2}{\sqrt{1-v_a^2/c^2}}$,

then $E_a = E_{a,k} + m_a c^2$.

Total energy of a single particle of mass m and velocity $\vec{v}$ depends on $v = |\vec{v}|$:

$E(v) = \dfrac{mc^2}{\sqrt{1-v^2/c^2}}$, the rest (that is, internal) energy equals

$E(0) = mc^2$, and kinetic energy can be computed as a difference:

$E_k(v) = E(v) - E(0)$. The formula $E(0) = mc^2$ is often called the <u>Einstein formula</u>.

Energy, momentum, and mass

Let us consider a particle of mass m, which in some IRF U has total energy E and momentum $\vec{p}$. Let us take the expression $E^2/c^2 - \vec{p}^2$ and let us try to put it as a function of the particle's velocity $\vec{v}$ (and mass):

$$\frac{E^2}{c^2} - \vec{p}^2 = \frac{m^2c^2}{1-v^2/c^2} - \frac{m^2v^2}{1-v^2/c^2} = m^2c^2.$$

It does not depend on the velocity $\vec{v}$, it depends only on the mass m, it is thus an invariant: its value is the same in all IRFs.
If in some other IRF U′ the same particle has total energy E′ and momentum $\vec{p}'$, then

$$\frac{E'^2}{c^2} - \vec{p}'^2 = \frac{E^2}{c^2} - \vec{p}^2 = m^2c^2.$$

The relationship between the total energy E of the particle, its momentum $\vec{p}$ and mass m, can also be written as

$$\underline{\underline{E^2 = c^2\vec{p}^2 + (mc^2)^2}} \quad \text{or} \quad \underline{\underline{E = \sqrt{c^2\vec{p}^2 + (mc^2)^2}}}.$$

It can be shown that the relationships between the quantities E/c, p_x, p_y, p_z (where p_x, p_y, p_z are components of the momentum vector $\vec{p}$) and E'/c, $p'_{x'}$, $p'_{y'}$, $p'_{z'}$ are the same as the relationships between spacetime coordinates $x^0 = ct$, x, y, z and $x'^0 = ct'$, x', y', z':

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{u}{c}p_x\right), \quad p'_{x'} = \gamma\left(p_x - \frac{u}{c}\frac{E}{c}\right), \quad p'_{y'} = p_y, \quad p'_{z'} = p_z, \qquad (*)$$

where $\gamma = 1/\sqrt{1-u^2/c^2}$. In other words, the quantities E/c, p_x, p_y, p_z transform as spacetime coordinates when one changes IRF. Any four quantities possessing this property form components of a certain four-vector:

$\left(\frac{E}{c}, p_x, p_y, p_z\right)$ are components of the energy-momentum four-vector of the particle,

whereas (ct, x, y, z) are components of the position four-vector of the particle (position with respect to the origin of the spacetime coordinate system).
Transformation rules (*) confirm that $E^2/c^2 - \vec{p}^2$ is invariant.

Let us compute the ratio $\underline{\frac{\vec{p}}{E} = \frac{\vec{v}}{c^2}}$. It has well defined limit when $|\vec{v}| \to c$:

$$\frac{|\vec{p}|}{E} \xrightarrow[|\vec{v}|\to c]{} \frac{1}{c}, \text{ that is } E \xrightarrow[|\vec{v}|\to c]{} |\vec{p}|c.$$

We therefore accept that particles moving at the speed of light (e.g. photons) obey the relation

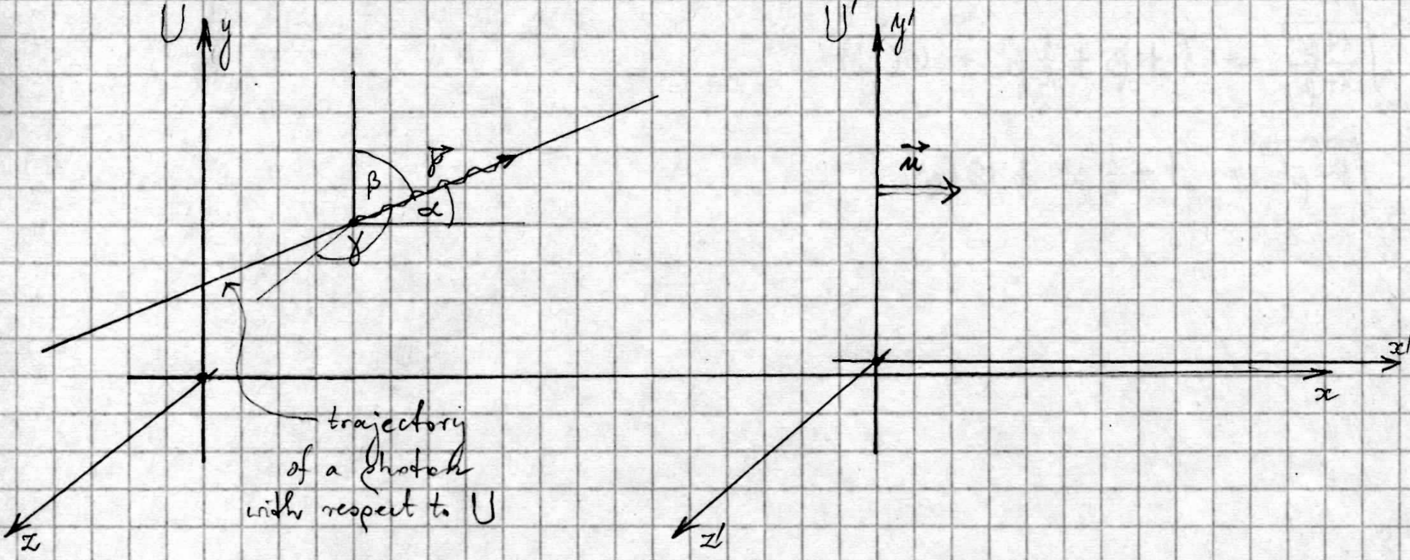
$$E = |\vec{p}|c.$$

It implies that then $m^2c^2 = E^2/c^2 - \vec{p}^2 = 0$, thus $m = 0$:

<u>the rest mass of the photon is equal to zero.</u>

√

The Doppler effect



α, β, γ — the angles between the momentum vector $\vec{p}$ of a photon and the three coordinate axes (cos α, cos β, and cos β are directional cosines, they fulfil $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$),

$p_x = |\vec{p}| \cos\alpha, \quad p_y = |\vec{p}|\cos\beta, \quad p_z = |\vec{p}|\cos\gamma;$

the photon's energy with respect to U: $E = |\vec{p}|c$,

and with respect to U′: $E' = |\vec{p}'|c$; E' and E are related through the equation:

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{u}{c}p_x\right)$$
$$= \gamma\left(\frac{E}{c} - \frac{u}{c}|\vec{p}|\cos\alpha\right)$$
$$= \gamma\frac{E}{c}\left(1 - \frac{u}{c}\cos\alpha\right);$$

Planck's constant

$E = h\nu, \quad E' = h\nu', \quad$ thus $\quad \nu' = \gamma\left(1 - \frac{u}{c}\cos\alpha\right)\nu,$

$$\nu = \frac{\sqrt{1-u^2/c^2}}{1-\frac{u}{c}\cos\alpha}\,\nu'.$$

Let the source of radiation be at rest in U′, then $\nu' = \nu_0$ is the radiation frequency in the proper frame of the source.

(i) The source is radially retreating with the speed u $(u>0)$:

$$\alpha = \pi, \quad \nu = \frac{\sqrt{1-u^2/c^2}}{1+\frac{u}{c}}\nu_0 = \sqrt{\frac{1-u/c}{1+u/c}}\,\nu_0 < \nu_0 \text{ — red-shift;}$$

(ii) the source is radially approaching with the speed u $(u>0)$:

$$u \longrightarrow -u, \quad \alpha = \pi, \quad \nu = \frac{\sqrt{1-u^2/c^2}}{1-u/c}\nu_0 = \sqrt{\frac{1+u/c}{1-u/c}}\,\nu_0 > \nu_0 \text{ — blue-shift;}$$

(iii) the motion of the photon is perpendicular to the x axis (the <u>transverse Doppler shift</u>):

$$\alpha = \frac{\pi}{2}, \quad \nu = \sqrt{1-u^2/c^2}\,\nu_0 < \nu_0.$$

$$\sqrt{\frac{1-\beta}{1+\beta}} = 1 - \beta + \frac{1}{2}\beta^2 + O(\beta^3)$$

$$\sqrt{\frac{1+\beta}{1-\beta}} = 1 + \beta + \frac{1}{2}\beta^2 + O(\beta^3)$$

$$\sqrt{1-\beta^2} = 1 - \frac{1}{2}\beta^2 + O(\beta^4)$$

Transformation of force

Let $\vec{F}$ be a force acting on a particle in some IRF U,
while $\vec{F}'$ is the same force acting on the same particle but measured in another IRF U',

$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}, \qquad \vec{F}' = F'_{x'} \vec{i}' + F'_{y'} \vec{j}' + F'_{z'} \vec{k}'.$$

We will find the relationship between the components F_x, F_y, F_z and $F'_{x'}, F'_{y'}, F'_{z'}$.

Equation of motion of the particle reads

$$\vec{F} = \frac{d\vec{p}}{dt} \text{ in } U, \qquad \vec{F}' = \frac{d\vec{p}'}{dt'} \text{ in } U'.$$

We compute:

$$F'_{x'} = \frac{dp'_{x'}}{dt'} = \frac{d[\gamma(p_x - \frac{u}{c}\frac{E}{c})]}{d[\gamma(t - \frac{u}{c^2}x)]} = \frac{dp_x - \frac{u}{c^2}dE}{dt - \frac{u}{c^2}dx} = \frac{\frac{dp_x}{dt} - \frac{u}{c^2}\frac{dE}{dt}}{1 - \frac{u}{c^2}\frac{dx}{dt}}$$

$$= \frac{F_x - \frac{u}{c^2}\frac{dE}{dt}}{1 - \frac{uv_x}{c^2}};$$

we employ the fact that $\frac{dE}{dt} = \frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}} = \vec{v}\cdot\frac{d\vec{p}}{dt} = \vec{v}\cdot\vec{F} = v_xF_x + v_yF_y + v_zF_z$,

we get

$$\underline{F'_{x'} = F_x - \frac{\frac{uv_y}{c^2}}{1-\frac{uv_x}{c^2}}F_y - \frac{\frac{uv_z}{c^2}}{1-\frac{uv_x}{c^2}}F_z;}$$

in a similar way we obtain

$$\underline{F'_{y'} = \frac{\sqrt{1-u^2/c^2}}{1-uv_x/c^2}F_y, \qquad F'_{z'} = \frac{\sqrt{1-u^2/c^2}}{1-uv_x/c^2}F_z.}$$

Let IRF U' be momentarily comoving with a particle,
then $v'_{x'} = v'_{y'} = v'_{z'} = 0$ and the following relations hold:

$$F_x = F'_{x'}, \quad F_y = \sqrt{1-\frac{u^2}{c^2}}\,F'_{y'}, \quad F_z = \sqrt{1-\frac{u^2}{c^2}}\,F'_{z'}.$$

✓

Relativistic equation of motion: another form

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\gamma\vec{v}\right) = m\left(\frac{d\gamma}{dt}\vec{v} + \gamma\frac{d\vec{v}}{dt}\right) = m\left(\frac{d\gamma}{dt}\vec{v} + \gamma\vec{a}\right),$$

$$m\frac{d\gamma}{dt} = \frac{\vec{v}}{c^2}\cdot\frac{d\vec{p}}{dt} = \frac{\vec{v}\cdot\vec{F}}{c^2},$$

$$\vec{F} = m\gamma\vec{a} + \frac{\vec{v}\cdot\vec{F}}{c^2}\vec{v},$$

$$\vec{a} = \frac{1}{m\gamma}\left(\vec{F} - \frac{\vec{v}\cdot\vec{F}}{c^2}\vec{v}\right);$$

in general $\vec{a}$ is not parallel to the force $\vec{F}$ acting on the particle.
Let us observe that for $|\vec{v}| \ll c$, $\vec{a} \cong \vec{F}/m$.

Motion in magnetic field

The magnetic force acting on a particle of mass m and velocity $\vec{v}$ moving in a magnetic field $\vec{B}$:

$$\vec{F} = q\,\vec{v}\times\vec{B}, \quad \text{where } q \text{ is electric charge of the particle.}$$

Relativistic equation of motion $\vec{a} = \frac{1}{m\gamma}\left(\vec{F} - \frac{\vec{v}\cdot\vec{F}}{c^2}\vec{v}\right)$ simplifies in this case, because

$$\vec{v}\cdot\vec{F} = q\,\vec{v}\cdot(\vec{v}\times\vec{B}) = 0, \quad \text{so}$$

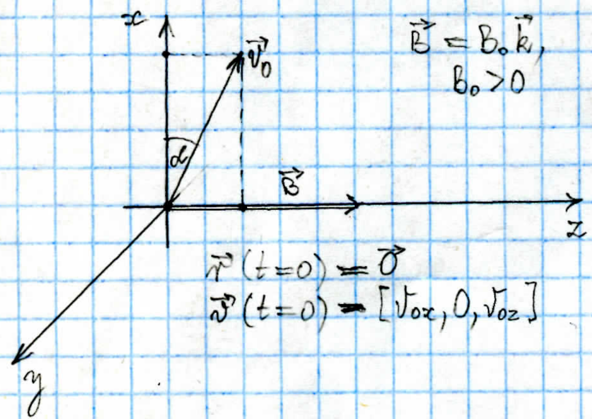
$$\vec{a} = \frac{\vec{F}}{m\gamma}.$$

During the motion in the magnetic field
<u>the magnitude of velocity is constant</u>, $|\vec{v}| = \text{const}$
(so the kinetic energy of the particle is constant as well, $E_k = \text{const}$):

$$|\vec{p}| = \sqrt{\vec{p}\cdot\vec{p}}, \quad \frac{d}{dt}|\vec{p}| = \frac{1}{|\vec{p}|}\,\vec{p}\cdot\frac{d\vec{p}}{dt} = \frac{1}{|\vec{p}|}(m\gamma\vec{v})\cdot(q\vec{v}\times\vec{B}) = \frac{mq\gamma}{|\vec{p}|}\,\vec{v}\cdot(\vec{v}\times\vec{B}) = 0,$$

so $|\vec{p}| = \text{const}$, and because $\vec{v} = (\vec{p}/m)/\sqrt{1+(\vec{p}/mc)^2}$, also $|\vec{v}| = \text{const}$.

Motion in uniform and constant in time magnetic field $\vec{B} = \text{const}$.



$\vec{B} = B_0\vec{k}$, $B_0 > 0$

$\vec{r}(t=0) = \vec{0}$
$\vec{v}(t=0) = [v_{0x}, 0, v_{0z}]$

$$\frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B} = q\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_0 \end{vmatrix} = q\left[\dot{y}B_0, -\dot{x}B_0, 0\right],$$

$$m\gamma\ddot{x} = q\dot{y}B_0, \quad m\gamma\ddot{y} = -\dot{x}B_0, \quad m\gamma\ddot{z} = 0.$$

$\ddot{z} = 0 \Rightarrow \underline{z(t) = v_{0z}t}$; $\quad \omega := qB_0/(m\gamma)$,

$\ddot{x} = \omega\dot{y}, \quad \ddot{y} = -\omega\dot{x}$

$\Downarrow$

$\dot{y} = -\omega x + C, \quad \dot{y}(0) = 0 = -\omega x(0) + C = C$, hence $\underline{\dot{y} = -\omega x}$

$\ddot{x} = \omega(-\omega x) = -\omega^2 x \Rightarrow x(t) = C_1\sin\omega t + C_2\cos\omega t,$

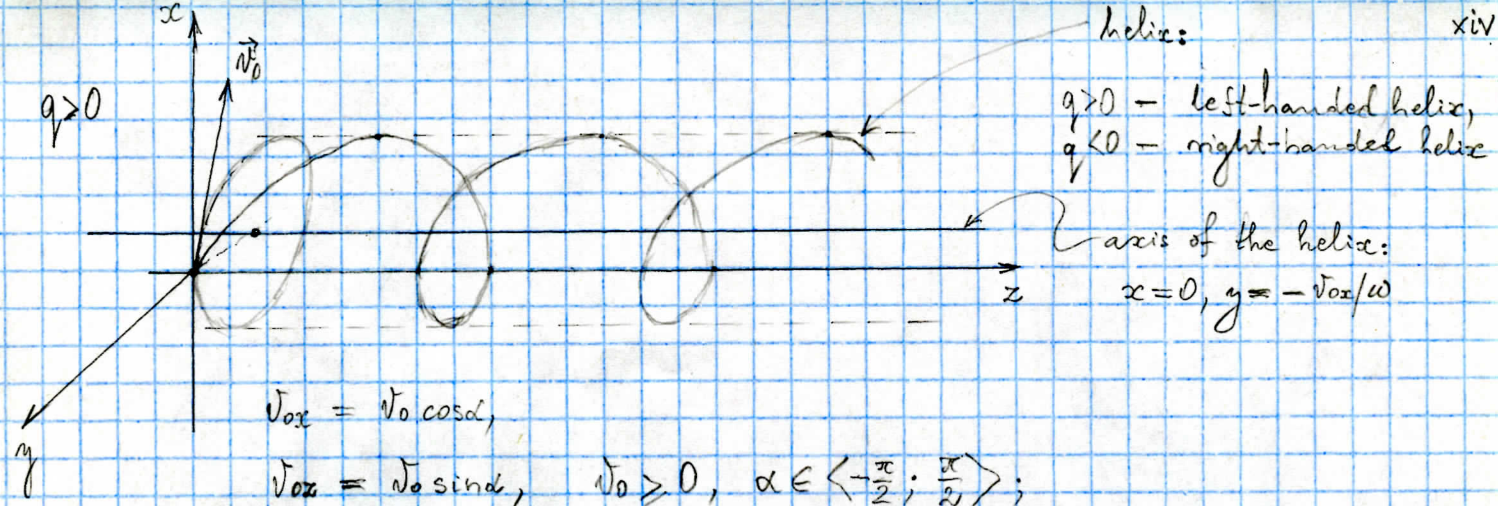
$x(0) = 0 = C_2, \quad x(t) = C_1\sin\omega t, \quad \dot{x}(t) = \omega C_1\cos\omega t,$

$\dot{x}(0) = \omega C_1 = v_{0x} \Rightarrow C_1 = v_{0x}/\omega,$

finally $\underline{x(t) = \frac{v_{0x}}{\omega}\sin\omega t}$;

$\dot{y} = -v_{0x}\sin\omega t \Rightarrow y = +\frac{v_{0x}}{\omega}\cos\omega t + C_3, \quad y(0) = \frac{v_{0x}}{\omega} + C_3 = 0 \Rightarrow C_3 = -\frac{v_{0x}}{\omega},$

$$\underline{y(t) = \frac{v_{0x}}{\omega}(\cos\omega t - 1).}$$



$v_{0x} = v_0 \cos\alpha,$

$v_{0z} = v_0 \sin\alpha, \quad v_0 \geq 0, \quad \alpha \in \langle -\frac{\pi}{2}; \frac{\pi}{2} \rangle;$

$$x(t) = \frac{v_{0x}}{\omega}\sin\omega t, \quad y(t) = \frac{v_{0x}}{\omega}(\cos\omega t - 1), \quad z(t) = v_{0z} t;$$

$x^2 + \left(y + \frac{v_{0x}}{\omega}\right)^2 = \left(\frac{v_{0x}}{\omega}\right)^2$ — a circle in (x,y) plane with center in $(0, -\frac{v_{0x}}{\omega})$ and radius $r = \left|\frac{v_{0x}}{\omega}\right|$,

$\underline{r = \frac{m v_0 \gamma \cos\alpha}{|q| B_0}}$, the projection of the particle onto the (x,y) plane runs along the circle with <u>cyclotron frequency</u> (gyrofrequency):

$\underline{\omega_c = |\omega| = \frac{|q| B_0}{m\gamma}}$, the period of the motion along the circle is $T_c = 2\pi/\omega_c = 2\pi \frac{m\gamma}{|q| B_0}$;

the <u>pitch</u> of the helix equals $v_{0z} \cdot T_c = 2\pi \frac{m v_0 \gamma \sin\alpha}{|q| B_0}$.

If $\underline{v_{0x} = 0}$ (i.e., $\alpha = -\frac{\pi}{2}$ or $\alpha = +\frac{\pi}{2}$), then $x(t) = y(t) = 0$, $z(t) = \mp v_0 t$:
the particle is at uniform motion along the axis z.

If $\underline{v_{0z} = 0}$ (i.e., $\alpha = 0$), then $x(t) = \frac{v_{0x}}{\omega}\sin\omega t$, $y(t) = \frac{v_{0x}}{\omega}(\cos\omega t - 1)$, $z(t) = 0$:
the particle is moving around the circle in the (x,y) plane.

Invariant mass of system of particles

Let us consider a system of n non-interacting particles. In an IRF U particles have total energies $E_1, E_2, \dots, E_n$ and linear momenta $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_n$. In another IRF U' their energies are $E_1', E_2', \dots, E_n'$, and their momenta equal $\vec{p}_1', \vec{p}_2', \dots, \vec{p}_n'$.

Let us introduce the index a to label different particles: the total energy of the a-th particle is E_a, its momentum equals $\vec{p}_a$, where $a = 1, 2, \dots, n$.

The relationship between E_a, $\vec{p}_a = (p_{ax}, p_{ay}, p_{az})$ and E_a', $\vec{p}_a' = (p'_{ax'}, p'_{ay'}, p'_{az'})$ results from the Lorentz transformation:

$$(*)\quad \begin{cases} E_a' = \gamma\,(E_a - u\,p_{ax}), \\ p'_{ax'} = \gamma\left(p_{ax} - \frac{u}{c^2}E_a\right), \qquad a = 1, \dots, n. \\ p'_{ay'} = p_{ay}, \quad p'_{az'} = p_{az}, \end{cases}$$

Let us define the total energy E and the total momentum $\vec{P}$ of the system of particles:

$$E := \sum_{a=1}^{n} E_a, \qquad \vec{P} := \sum_{a=1}^{n} \vec{p}_a, \quad \text{so} \quad \vec{P} = (P_x, P_y, P_z),$$

$$\text{where } P_x = \sum_{a=1}^{n} p_{ax}, \quad P_y = \sum_{a=1}^{n} p_{ay}, \quad P_z = \sum_{a=1}^{n} p_{az}.$$

Analogous formulas apply in IRF U'.

By adding individual equations from the system $(*)$ for all $a = 1, \dots, n$ one gets

$$(**)\quad \begin{cases} E' = \gamma\,(E - u\,P_x), \\ P'_{x'} = \gamma\left(P_x - \frac{u}{c^2}E\right), \\ P'_{y'} = P_y, \quad P'_{z'} = P_z. \end{cases}$$

Equations $(**)$ imply that if in IRF U energy and momentum of the system are conserved ($E = \text{const}$, $\vec{P} = \text{const}$), then energy and momentum are also conserved in IRF U' ($E' = \text{const}$, $\vec{P}' = \text{const}$) as long as both of these conservation principles hold simultaneously. This indicates that both conservation principles are not independent of each other.

Equations $(**)$ also imply that $E^2 - c^2\vec{P}^2 = E'^2 - c^2\vec{P}'^2 = \text{const}$, therefore one can define an invariant mass M of the system of particles with the aid of equation

$$E^2 - c^2\vec{P}^2 = M^2c^4,$$

$$\text{so} \quad M = \frac{1}{c^2}\sqrt{E^2 - c^2\vec{P}^2} = \frac{1}{c^2}\sqrt{\left(\sum_{a=1}^{n} E_a\right)^2 - c^2\left(\sum_{a=1}^{n} \vec{p}_a\right)^2}.$$

Center-of-mass reference frame

Center-of-mass (COM) reference frame is defined as such IRF in which the total momentum of the system vanishes. Velocity of COM frame with respect to any IRF U equals
$$\vec{V}_{COM} = \frac{c^2\vec{P}}{E}.$$

Let the vector $\vec{P}$ be directed along the x axis: $\vec{P} = (P, 0, 0)$, where $P = |\vec{P}|$, then $P_x = P$, $P_y = P_z = 0$. Let the IRF U' moves with respect to U with velocity
$$\vec{u} = \vec{V}_{COM} = \left(\frac{c^2P}{E}, 0, 0\right).$$

Then the total momentum $\vec{P}'$ of the system in IRF U' equals
$$P'_{x'} = \gamma\left(P_x - \frac{u}{c^2}E\right) = \gamma\left(P - \frac{c^2P}{E}\frac{E}{c^2}\right) = 0, \quad P'_{y'} = P_y = 0, \quad P'_{z'} = P_z = 0,$$
so IRF U' is COM frame.

The position of the COM of the system of non-interacting particles is defined as
$$\vec{R}_{COM} = \frac{\sum_{a=1}^{n} E_a\vec{r}_a}{\sum_{a=1}^{n} E_a}.$$

This is so because
$$\frac{d\vec{R}_{COM}}{dt} = \frac{1}{E}\sum_{a=1}^{n} E_a\vec{v}_a = \frac{c^2}{E}\sum_{a=1}^{n} m_a\gamma_a\vec{v}_a = \frac{c^2}{E}\vec{P} = \vec{V}_{COM}.$$

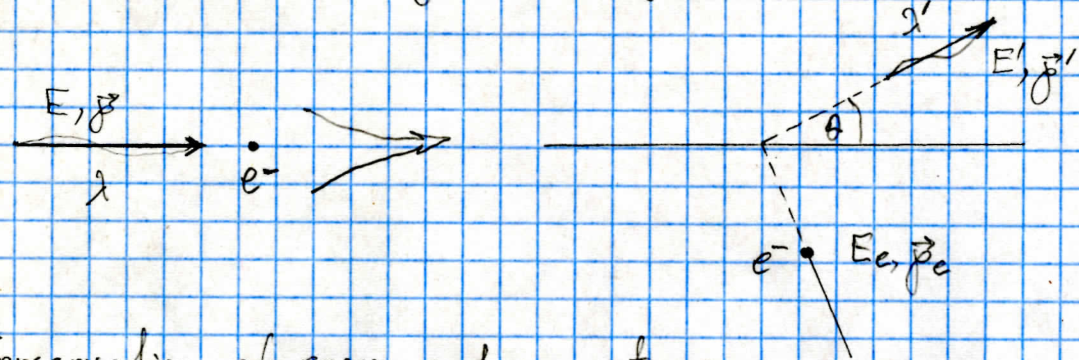
In non-relativistic approximation:
$$E_a = m_a c^2 + E_{ak} \cong m_a c^2, \quad \text{so} \quad \vec{R}_{COM} \cong \frac{\sum_{a=1}^{n} m_a c^2\vec{r}_a}{\sum_a m_a c^2} = \frac{\sum_{a=1}^{n} m_a\vec{r}_a}{\sum_a m_a}.$$

In COM reference frame the invariant mass $\mathcal{M}$ of the system equals its total energy divided by c^2:
$$\mathcal{M} = \frac{E}{c^2} = \frac{1}{c^2}\sum_{a=1}^{n} E_a.$$

√

Compton scattering

relies on scattering of X rays on electrons.



Conservation of energy and momentum:

$$\begin{cases} \vec{p} = \vec{p}\,' + \vec{p}_e, \\ E + m_e c^2 = E' + E_e, \end{cases}$$

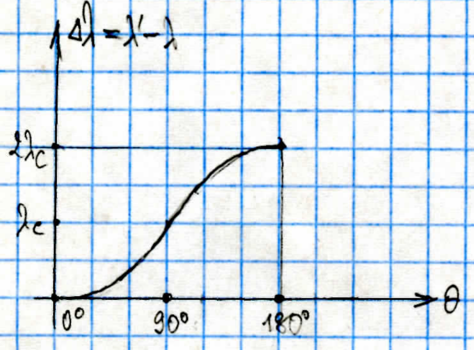
$$E = |\vec{p}|c, \quad E' = |\vec{p}\,'|c, \quad E_e = \sqrt{c^2 |\vec{p}_e|^2 + m_e^2 c^4}$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2}(1-\cos\theta),$$

$$E = \frac{hc}{\lambda}, \quad E' = \frac{hc}{\lambda'},$$

$$\underline{\underline{\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta) = \lambda_C (1-\cos\theta),}}$$

where $\lambda_C := \frac{h}{m_e c} \doteq 0.0243$ Å — <u>Compton's wavelength of electron.</u>



Particle accelerators: head-on beams

Let us consider collision of two protons (e.g. in the LHC accelerator):

$$p + p \to X + Y + \cdots .$$

What fraction of the energy of the colliding protons can be used to produce new particles X, Y, ...? (This is called available energy.)

Let us consider the collision in the center-of-mass reference frame, in which conserved linear momentum vanishes:

$$\underbrace{\vec{p}_1^{\,com} + \vec{p}_2^{\,com}}_{\text{before collision}} = \vec{0} = \underbrace{\vec{p}_X^{\,com} + \vec{p}_Y^{\,com} + \cdots}_{\text{after collision}},$$

conservation of energy reads

$$\underbrace{E_1^{com} + E_2^{com}}_{\text{before collision}} = \underbrace{(m_X + m_Y + \cdots)c^2 + (\text{kinetic energies})}_{\text{after collision}}.$$

In the center-of-mass frame the whole energy of the colliding protons can be changed into the internal energy of particles created in the collision. If this is the case, then

$$E_1^{com} + E_2^{com} = (m_X + m_Y + \cdots)c^2$$ and all particles X, Y, ... are at rest in the COM frame.

Thus $\underline{\Delta E^{com} = E_1^{com} + E_2^{com}}$ is the amount of energy which can be used for production of new particles in the COM frame.

Let us now consider the same collision in reference frame, in which one of the protons is at rest (we call this laboratory frame for the time being). How large the energy of the second proton should be, to ensure that the energy available in laboratory frame for production of new particles will be equal to ΔE^{com}?

$$\Delta E^{com} = E_1^{com} + E_2^{com} = \sqrt{\underbrace{(E_1^{com} + E_2^{com})^2 - c^2(\vec{p}_1^{\,com} + \vec{p}_2^{\,com})^2}_{\substack{\text{invariant of Lorentz transformation} \\ [\text{equal to } c^2 \times (\text{invariant mass})]}}}$$

$$= \sqrt{(E^{lab} + m_p c^2)^2 - c^2 (\vec{p}^{\,lab})^2}$$

$$= \sqrt{2 m_p c^2 (E^{lab} + m_p c^2)} \implies \underline{E^{lab} = \frac{(\Delta E^{com})^2}{2 m_p c^2} - m_p c^2}.$$

In LHC protons are moving head-on (so laboratory frame is also center-of-mass frame) with energies $E^{com} \cong 7$ TeV ($1\ \text{TeV} = 10^{12}\ \text{eV} = 1000\ \text{GeV}$), so $\Delta E^{com} = 2E^{com} \cong 14$ TeV.

The same available energy can be achieved by colliding with the proton at rest another proton with energy

$$\frac{(\Delta E^{com})^2}{2 m_p c^2} - m_p c^2 \cong \frac{(\Delta E^{com})^2}{2 m_p c^2} \cong 10^5\ \text{TeV} \quad (m_p c^2 \cong 938.3\ \text{MeV}; \ 1\,\text{eV} = 1e \cdot 1\text{V} \cong 1.602 \times 10^{-19}\ \text{J}).$$

Particle - antiparticle pair creation

Let us consider creation of electron-positron pair by a γ photon:

$$\gamma \longrightarrow e^- + e^+ .$$

Conservation of energy implies, that this reaction could happen provided $E_\gamma > 2 m_e c^2 \cong 1.02$ MeV. But this reaction can not happen in vacuum, because it would violate conservation of momentum:

in the center-of-mass frame of electron-positron pair $\vec{p}_{e^+} + \vec{p}_{e^-} = \vec{0}$,

but $\vec{p}_\gamma \neq \vec{0}$ — in all reference frames photon's momentum is non-zero.

Creation of electron-positron pair can happen in the vicinity of another particle, e.g. nucleus of some atom.